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# PARAMETRIC OPTIMAL DESIGN

## *Part 1: First Order Algorithm*

by B.M. KWAK, K. RIM and J.S. ARORA

MARCH 1975

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PARAMETRIC OPTIMAL DESIGN

PART I: FIRST ORDER ALGORITHM

by

B. M. Kwak, K. Rim, and J. S. Arora

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## LIST OF SYMBOLS

$A$	free parameter ( $\alpha$ ) constraint set; partial derivatives of constraint function with respect to free parameter vector ( $\alpha$ )
$A(x)$	answering set at $x$ of a function $\phi(x, \alpha)$
$\overline{AA}$	second partial derivative of constraint function with respect to free parameter vector ( $\alpha$ )
$B$	a Banach space; partial derivatives of constraint function with respect to design variable vector ( $b$ )
$B^*$	the normed dual of $B$
$\overline{BA}$	cross partial derivative of constraint function with respect to design variable and free parameter vector
$\overline{BB}$	second partial derivative of constraint function with respect to design variable vector ( $b$ )
$b$	design variable vector
$-C$	a vector in the first order feedback law compensating the error in current maximum point
$C(A)$	a normed function space with domain $A$
$D$	constraint region
$-D$	sensitivity coefficient of the maximum point, as given in the first order feedback law
$E$	Young's modulus
$e$	direction vector
$F(\cdot), f(\cdot)$	functions
$G(\cdot), g_1(\cdot)$	functions

$h_1(\cdot)$	function associated with state equations
$I$	identity matrix; moment of inertia of a beam when used after $E$ ; index set
$I_\epsilon$	$\epsilon$ -active index set
$J$	objective function
$K^\phi$	constant matrix for constraint functions due to the vector $C$
$L^J$	second derivative of objective function with respect to design variable vector
$l^J$	first derivative of objective function with respect to design variable vector
$L^\phi$	matrix of second derivatives of constraint functions with respect to design variable vector
$l^\phi$	matrix of first derivatives of constraint function with respect to design variable vector
$M$	tangent subspace
$M(x)$	set of support functionals to a function at $x$ ; moment along the beam axis
$M, M_q, M_{qq}$	various matrices associated with free parameter constraint set
$M_{\phi\phi}$	matrix associated with design variable constraint set
$m$	dimension of state variable vector ( $z$ )
$n$	dimension of design variable vector ( $b$ )
$O(x)$	$O(x)/  x   \leq K$ for small $  x  $ and some constant $K$
$p$	dimension of free parameter vector ( $\alpha$ )
$Q$	matrix of gradients for free parameter constraint equations
$q_1(\alpha)$	function associated with free parameter constraint set

$R^n$	n dimensional Euclidean space
$r$	number of free parameter constraint equations
$S(\cdot)$	algorithm
$U_1$	neighborhood
$U_1(x)$	answering set at $x$ on $U_1$ for $\phi(x, \alpha)$
$W$	positive definite weighting matrix
$W_b$	weighting matrix associated with design variables
$W_a$	weighting matrix associated with free parameters
$x, y$	vectors in a Euclidean space
$z$	state variable vector
$\alpha$	free parameter vector
$\beta$	small parameter for implementation of algorithm
$\gamma$	Lagrange multiplier for design variable step size constraint
$\Gamma_x$	set of feasible directions at $x$
$\delta(\cdot)$	variation of $(\cdot)$
$\Delta(\cdot)$	desired reduction in $(\cdot)$
$\epsilon, \epsilon_0, \epsilon_A, \epsilon_B, \epsilon_S$	small parameters for implementation of algorithm
$\eta$	constant for design variable stepsize restriction
$\kappa$	adjoint variable vector to the linearized state equations
$\lambda, \mu$	Lagrange multiplier vectors
$q_\lambda$	Lagrange multiplier for inequalities in second order approximate problem
$\mu(\cdot)$	function



$\nu$	Lagrange multiplier for free parameter step size constraint
$\xi$	constant for free parameter stepsize restriction
$\rho$	density
$\sigma$	stress
$\sigma_Y, \sigma_T, \sigma_C$	yield stresses
$\phi(\cdot)$	function
$\Phi$	modified matrix of second partial derivatives of parametric constraint functions with respect to free parameter vector
$\omega$	excitation frequency

### Superscripts

$(1), 1$	1-th iterate
$1, 2$	decomposition of a vector by two components
$T$	transpose of a matrix

### Subscripts

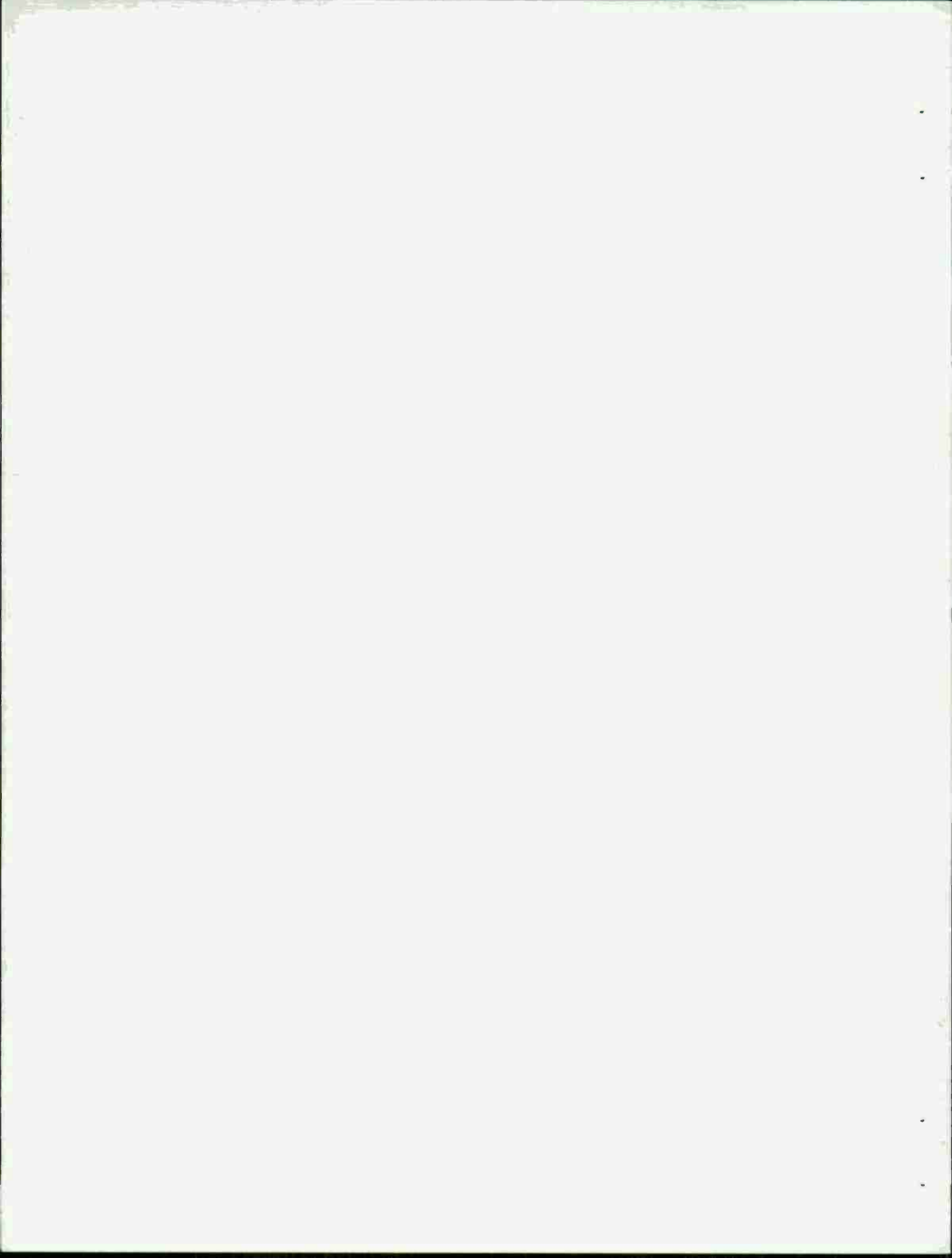
$i$	1-th component of a vector
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### Miscellaneous Symbols

max	maximize
min	minimize
NLP	nonlinear programming
POD	parametric optimal design
sup	supremum
$\epsilon$	belong to
$\nabla(\cdot)$	gradient



- ( $\vec{\cdot}$ )      solution vector; a quantity calculated at the solution
- $||\cdot||$       norm (usually Euclidean norm)
- $\cup$       union
- [ ]      denotes reference numbers listed in REFERENCES



## CHAPTER 1

### INTRODUCTION

#### 1.1 Motivation, Contributions of the Thesis, and Organization

The goal of engineering can be viewed as synthesis rather than analysis for a desired system. The concept of optimization arises naturally in engineering as an indispensable tool of conceptualization and synthesis

Most formulations of optimal design problems in engineering assume that complete information is given, in the sense that the system is operated under a certain known condition. The formulation, then, generally reduces to a "nonlinear programming" problem. In reality, however, few systems operate without interaction with their environment. A mechanical system, for example, must operate over a range of load and environmental parameters. Applied loads on a structure may act over a range of positions and angular orientations, at every point of which strength and displacement constraints must be satisfied. Operating temperature and excitation frequencies are other examples of environmental parameters that vary

in known intervals, over which performance constraints must be satisfied. Military systems, for example, may be required to operate satisfactorily from  $-40^{\circ}\text{F}$  to  $120^{\circ}\text{F}$ . One cannot simply optimize for a given temperature and accept the performance that occurs at other temperatures. Rather, performance constraints must be satisfied at all temperatures.

To include the effect of surrounding systems, a new formulation is necessary. The contribution of this thesis is to define a new problem formulation and to develop numerical algorithms to solve this class of optimization problems. The development is solution-oriented, instead of being a theoretical characterization of the solution of the problem. The basic approach rests on the idea of making a series of approximate problems out of the original problem, by use of first and second variations, making it possible to compute solutions to practical problems in an iterative fashion.

Following a literature survey of topics and problems, similar in nature to the present concept of problem formulation, a precise definition of the problem is given in Chapter 2. This is followed by a review of certain classical problems and their relation to the present formulation. Several concrete example problems, which are solved in subsequent chapters, are then stated to make it clear

what the mathematical formulation implies. The mathematical background relevant to the development of computational algorithms is given at the end of Chapter 2.

The remainder of the thesis develops two computational algorithms and presents numerical solutions of example problems. Chapter 3 presents a first order algorithm, with four example problems solved. It is found that this algorithm is simple enough to be implemented without excessive computational or analytical complexity and, when it is applied to min-max problems, it gives a justification for alternating maximization and minimization procedures.

The main contribution of Chapter 4 lies in sensitivity analysis with error compensation, which makes second order algorithms and other interesting hybrid methods of Chapter 5 possible. Several example problems are solved, using both the second order algorithm and the hybrid method, and a comparison is made.

Since the problem formulation itself is rather complicated, the algorithms are algebraically complex. Implementation of the algorithms is given in Chapter 6, where a general flow chart is described.

Finally, the last chapter provides a summary of the main contributions and the difficulties involved in application of the developed algorithms.

## 1.2 Literature Survey

Several mathematical programming formulations that involve a free parameter have been considered in the literature. Fritz John [1] considered an optimization problem with a continuum of constraints and showed that the problem can be characterized by one having a number of constraints less than or equal to the dimension of the design variable. Recently, Gehner [2] considered the same problem, with a continuum of equality constraints, and derived a necessary condition that is an extension of the Fritz John theorem. He also studied constraint qualification to Chebyshev approximation.

Minimax theory, which is closely related to the worst case design problem, has been studied by many authors and has been applied in fields such as matrix game theory, optimal control theory, and differential games. For a review of the literature pertaining to minimax problems, the reader is referred to McLinden [3] and Barry [4]. Recently min-max problems have received considerable attention. Danskin [5] studied the max-min problem, using the directional derivative of the function  $\phi(x) = \min_y f(x,y)$ , and examined several military applications to resource allocation. Pshenichnyi [6] gives an extensive study of necessary conditions for such extrema, in a general setting. He treats functions of the type  $\phi(x)$ , above, and applies results to the mathematical

programming problem with a continuum of constraints.

For solutions of the minimax problem, Heller [7] uses the directional derivative in an iterative scheme and defines a minimax gradient direction. Medanic [8] developed an elimination algorithm in which he contracts the feasible set and decreases the minimax objective function. In this treatment, the minimax objective function was assumed to be convex in the set over which minimization is carried out.

Bracken and McGill [9] consider a mathematical programming problem with optimization problems in the constraints. They develop criteria under which the program is convex.

The problem that is treated in this thesis differs from those considered above, in the sense that it is optimal design oriented. Little emphasis is given in the literature on numerical algorithms to solve the problem cited above. The main emphasis of the present thesis is on the development of numerical algorithms for the computation of solutions to practical problems.



## CHAPTER 2

### STATEMENT OF THE PROBLEM

#### 2.1 Problem Definition

##### 2.1.1 Parameters in the Optimal Design Problem

The process of optimal design is that of synthesizing a system for the stated needs and objectives. A physical system is modeled so that mathematical analysis is possible and the cost function that is to be minimized is analytically defined. In this section, certain parameters of the system are grouped to provide a model of an optimal design problem. Three groups are identified, according to their role in the given system and in representing the environment. They are: design variables of the system, parameters associated with the interacting environment, and parameters associated with the state of the system. These will be called "design variables," "environmental parameters" or "free parameters," and "state variables," respectively. Design variables are those quantities that are to be determined by the designer to specify the design of the system. Environmental parameters are quantities that represent the environment. The designer will not know the exact condition of the



environment, which can change over the life of the system. Hence, from the designer's point of view, environmental parameters are uncertain quantities. State variables are those quantities that characterize the state of the system due to a given environment. The state of the system is determined by a set of equations, called state equations, once the design variables and environmental parameters are fixed. The process of solving the state equations is the analysis part of design synthesis.

This classification of variables is not necessarily unique, but is dependent on the designer's point of view. As a particular example, consider the simple vibration system shown in Figure 2. The mass of each rigid body, the geometry of the system, the properties of the spring and damper, etc., are the design variables of the system. The frequencies of the excitation forces and their magnitudes or directions are considered as environmental parameters. The position, velocity, and acceleration of each mass are the state variables. An example of a more complicated problem is found in [5].

In the finite dimensional optimal design problem, the three groups of parameters are represented as vectors in suitably chosen vector spaces. The vectors will be denoted by  $b$ ,  $\alpha$ , and  $z$ ; for design variables, environmental parameters, and state variables, respectively. The

dimensions of these vectors will be denoted by  $n$ ,  $p$ , and  $m$ , respectively.

### 2.1.2 Mathematical Statement of the Problem

After the various parameters for a given system are defined, one is in a position to state the design problem in mathematical terms. The mathematical model for the class of optimal design problems introduced here will be termed finite dimensional "parametric optimal design," or in short POD. It may be stated as follows: Minimize

$$J = f(b) \quad (2.1)$$

subject to the state equations

$$h_1(z, b, \alpha) = 0, \quad i=1, \dots, m; \quad (2.2)$$

the parametric constraints

$$\max_{\alpha \in A} g_\beta(z, b, \alpha) \leq 0, \quad \beta=1, \dots, l; \quad (2.3)$$

and the design variable constraints

$$g_\beta(b) \leq 0, \quad \beta=l+1, \dots, l', \quad (2.4)$$

where

$$A \equiv \{\alpha \mid q_1(\alpha) \leq 0, \quad i=1, \dots, r\} \quad (2.5)$$

will be called the  $\alpha$ -constraint set.

It is noted that certain of the constraints in the POD formulation may be absent for a specific problem, as shown by examples in Section 2.3. The special feature of the formulation lies in the inclusion of parametric constraints with state equations. Otherwise, the problem

reduces to a nonlinear programming problem. The generality of the POD formulation is amplified in Section 2.2, in relation to other classes of problems.

### 2.1.3 Subproblems and Expansion Procedures

For the analytical development to follow, subproblems are defined as the maximization problems given in the parametric constraints of Eq. (2.3), i.e.,

$$\max_{\alpha \in A} g_{\beta}(z, b^{(0)}, \alpha) \quad (2.6)$$

subject to

$$h_i(z, b^{(0)}, \alpha) = 0, \quad i=1, \dots, m, \quad (2.7)$$

where  $b^{(0)}$  is an assumed design.

Throughout the paper, it is assumed that the implicit function theorem can be applied to the state equations, i.e.,  $\partial h / \partial z$  is nonsingular and  $z$  is determined locally as a continuous function of  $b$  and  $\alpha$ . It is also assumed that  $g_{\beta}$  is continuous with respect to its arguments and  $\partial g_{\beta}(z(b, \alpha), b, \alpha) / \partial b$  is continuous with respect to  $b$ . To ensure the existence of the solution to the subproblem, the set  $A$  is assumed to be closed and bounded, hence compact (Section 2.4). For a physically well formulated problem, these assumptions are not particularly restrictive.

For further developments, it is assumed that the subproblem has a countable, preferably finite, number of solution points. The underlying idea is that, for each

local maximum point of the subproblem, a small closed neighborhood is considered and an approximate subproblem is constructed with the set  $A$  replaced by this neighborhood. Thus, the newly generated subproblems, called inner problems, replace the original subproblems, expanding the number of parametric constraints. Formally, the parametric constraints can be written as

$$\max_{\alpha \in U_1} g_{\beta_1}(z, b^{(0)}, \alpha) \leq 0, \quad (2.8)$$

where  $U_1 \subset A$  is a closed neighborhood around the local maximum point  $\alpha_1^{(0)}$  that is large enough to contain the neighboring maximum point  $\alpha_1^{(1)}$ , after a small change in  $b^{(0)}$  to  $b^{(1)}$ , but small enough to contain only one local maximum point. Under these conditions, it will be shown in Section 2.4 that the function on the left side of Eq. (2.8) is differentiable with respect to  $b$ .

It will be assumed that the expansion procedure has been applied for each numerical algorithm that follows. Hence, the words "inner problem" and "subproblem" will be used interchangeably.

## 2.2 Relation to Other Classes of Problems

The POD problem formulation generalizes a number of classical problems. In this section, the relation to these problems is studied.

### 2.2.1 Min-Max Problem and Game Theory

In zero-sum two-person game theory, a mixed strategy is sought such that the max-min is equal to min-max, by randomization [11,12,13]. Game theory is not the subject of this thesis, but if one knows that a mixed strategy exists, one can obtain a solution by solving the min-max problem only. In general, the min-max problem; i.e., a two stage problem corresponding to "half" a saddle point problem; arises from two stage resource allocation problems [5].

"Worst case design" can be considered as a game problem between the designer and nature, and can be formulated as a min-max problem. The designer assumes that nature will pick the environmental parameter by maximizing the cost, whatever design the designer chooses. The design objective is that the designer minimizes this worst cost, or loss ceiling. This design concept is very conservative. One may mitigate this conservativeness by introducing a suitable probability distribution over the environmental parameters. This process of mitigation does not disturb the min-max nature of the physical design problem.

Since this thesis is directed toward a solution method, the min-max problem is assumed formulated. In this case, the POD formulation is realized through introduction of a dummy design variable,  $b_{n+1}$ , called an artificial



design variable such that the transformed problem is to minimize

$$b_{n+1} \quad (2.9)$$

subject to the constraint

$$\max_{\alpha \in A} \phi(b, \alpha) \leq b_{n+1}. \quad (2.10)$$

Although the dimension of the design variable is increased by 1 and an additional subproblem must be treated, this approach has the advantage that every possible local maximum point of  $\phi(b, \alpha)$ , with respect to  $\alpha$ , can be retained by expansion of the number of constraints, as in Section 2.1. It is also noted that a cost function of the form  $\max_{\alpha \in A} \phi(z, b, \alpha)$  can be transcribed by the same technique. Another notable application may be the saddle point statement for the NLP problem given in Section 2.4, which can be treated formally by the POD algorithm.

### 2.2.2 Chebyshev Approximation

The theory of approximation was developed independently of optimization theory, especially the theory of Chebyshev approximation. Literature on approximation theory is voluminous, for example, see [14-22].

Suppose a function  $f(\alpha)$  of some normed space  $C(A)$  is given, and further an approximating function  $F(b, \alpha)$  depending on the parameter  $b \in B$  where  $B$  is some domain set of parameter  $b$  is given. Then a general approximation

problem is: Given  $f \in C$ , determine the parameter  $b^0 \in B$  such that

$$||f - F(b^0)|| \leq ||f - F(b)|| \quad (2.11)$$

for all  $b \in B$ . Here the norm on  $C$  is denoted by  $||\cdot||$ .

If the norm is given by the uniform norm  $||f|| = \max_{\alpha \in A} |f(\alpha)|$ ,

the problem is called Chebyshev approximation and can be

restated as: Determine a  $b^0 \in B$  to minimize

$$\max_{\alpha \in A} [g(b, \alpha) \equiv |f(\alpha) - F(b, \alpha)|]. \quad (2.12)$$

Usually the set  $\{F(b, \alpha) | b \in B\}$  is a subset of  $C$ . If  $F(b, \alpha)$  is a linear functional on  $B$  for each fixed  $\alpha$ , the approximation problem is called linear, otherwise nonlinear. Since all the required values of the components of  $b$  are necessarily finite, the set  $B$  can be chosen to be bounded. Furthermore, if  $B$  is assumed closed and  $F(b, \alpha)$  is continuous in the parameters, there is a solution [17].

The study of approximation problems through optimization techniques is still in its infancy, especially in the field of nonlinear Chebyshev approximation. In the case of discrete Chebyshev approximation, a finite number of discrete points are chosen in the given region and the problem is relatively simple [14, 23, 24]. In the case of continuous approximation, such as polynomial and rational approximations, a few algorithms were developed by Stiefel [22] and Remes [16, 19]. These algorithms are based on the theorem of Chebyshev and are algebraic in nature. Recent

study of Chebyshev approximation, through optimization theory [2,6,25], gives necessary conditions that are similar to the Chebyshev theorem, but do not provide a constructive method of solution.

Chebyshev approximation falls into the present formulation of POD, hence the algorithm developed here can be applied easily, regardless of whether the approximation is linear or nonlinear.

### 2.2.3 NLP with State Equations Given Implicitly by a Minimum Principle

Using sensitivity analysis, which will follow in Chapter 4, an efficient reanalysis of the state equations is possible, particularly if the state equations can be derived from a minimum principle. The problem is stated in the form: Minimize

$$f(b,z) \quad (2.13)$$

subject to the constraints

$$g(b,z) \leq 0, \quad (2.14)$$

where  $z$  solves the nonlinear programming problem

$$\min_{z \in \Omega} H(b,z) \quad (2.15)$$

where  $\Omega$  is a constraint set.



### 2.3 Examples to be Considered

To visualize the type of problems dealt with in this thesis, several concrete example problems are stated. After solution methods are developed, these problems are solved numerically and a comparison is made with results in the literature.

#### 2.3.1 Problem of Finite Allocation

The problem considered here is a finite allocation game. This problem is discussed in [5,8]. Suppose a mix  $x = (x_1, \dots, x_n)$  has to be chosen among a set of  $n$  weapon systems, given a total budget  $C_b$ , that are to be used in the possible attack from  $n$  enemy units. Whatever attack the enemy selects the damage inflicted on the enemy from the counterattack is to be maximized. That is, the enemy will strike first with full knowledge of the system mix  $x$ . If the  $i$ -th weapon system can not attack the  $j$ -th enemy target, or vice versa, the residual  $x_1'$  after enemy attack of strength  $y_1$  can be assumed given by the formula:

$$x_1' = x_1 e^{-\gamma_1 y_1}$$

where  $\gamma_1$  is the "vulnerability" of  $i$ -th weapon system, with respect to  $i$ -th enemy attack. This residual is used in the counterattack. The damage to the  $i$ -th enemy target is given by

$$v_1(1 - e^{-\beta_1 x_1'}),$$

where  $v_1$  is the value of the 1-th target and  $\beta_1$  is the vulnerability of the 1-th enemy target. Supposing the total damage is obtained by addition of individual system damage, it is

$$\phi(x, y) = \sum_{i=1}^n v_i (1 - e^{-\beta_i x_i} e^{-\gamma_i y_i}).$$

Since  $y_1$  is chosen in full knowledge of  $x_1$ , optimal choice for  $x_1$  is obtained by maximizing  $\min_y \phi(x, y)$ ; or equivalently by minimizing  $\max_y [-\phi(x, y)]$ , with the budget restrictions

$$\begin{aligned} \sum x_1 &= C_b, \\ \sum y_1 &= C_a, \\ x_1 &\geq 0, \text{ and } y_1 \geq 0. \end{aligned}$$

The two equality constraints above are used to reduce the dimension of the variables by removing  $x_n$  and  $y_n$ . Here  $x_1$  plays the role of a design variable,  $b_1$ , and  $y_1$  plays the role of the environmental parameter,  $\alpha_1$ . The allocation problem is formulated as a POD problem as: Minimize

$$J \equiv \max_{\alpha} [f' = \sum_{i=1}^n v_i (T_i - 1)] \quad (2.16)$$

subject to

$$C_b - \sum_{i=1}^{n-1} b_i \geq 0, \quad (2.17)$$

$$b_i \geq 0, \quad i=1, \dots, n-1,$$

$$C_a - \sum_{i=1}^{n-1} \alpha_i \geq 0, \quad (2.18)$$

$$\alpha_i \geq 0, \quad i=1, \dots, n-1,$$

where

$$T_1 = e^{-\beta_1 b_1} e^{-\gamma_1 \alpha_1}$$

and

$$T_n = e^{-\beta_n (C_b - \sum_{i=1}^{n-1} b_i)} e^{-\gamma_n (C_\alpha - \sum_{i=1}^{n-1} \alpha_i)} \quad (2.19)$$

This formulation is transformed to fit the POD formulation by introducing an artificial design variable, as explained in Section 2.2.

### 2.3.2 Three Bar Truss Design Problem

For the truss shown in Figure 1, one wishes to determine the cross sectional area of each member in order to minimize the weight of the structure, under constraints (stress, buckling, and displacement) that must be satisfied for any load direction  $\alpha$  in a given range,  $\alpha_1 \leq \alpha \leq \alpha_2$ .

The same problem, without parametric constraints, has been used as a test problem for single or multiple load conditions [26,27,28]. The problem with stress parametric constraints is discussed in [29], in which the author uses the "integrated penalty function."

Denoting the cross sectional areas as  $b_i$ ,  $i=1,2,3$ , and defining moments of inertia by  $I_i = \beta b_i^2$ , one may state the POD problem as: Minimize

$$f = 10\sqrt{2} b_1 + 10 b_2 + 10\sqrt{2} b_3, \quad (2.20)$$

subject to stress constraints

$$g_1 = 5 \cdot 10^5 |z_1 + z_2| - 5 \cdot 10^3 \leq 0, )$$

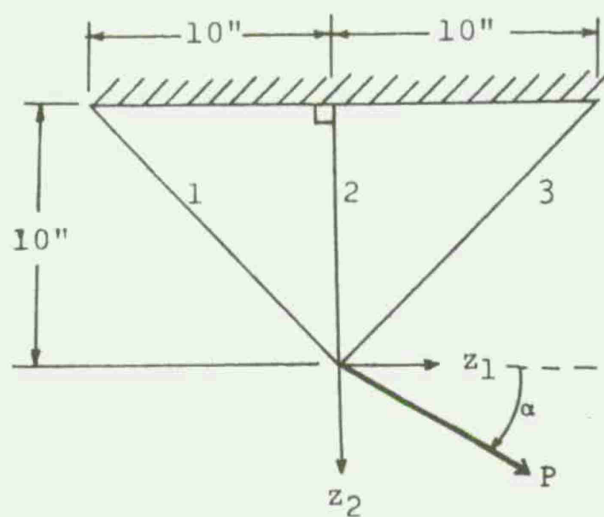


Figure 1 A Three Bar Truss

$$\left. \begin{aligned} g_2 &\equiv 10^6 |z_2| - 2 \cdot 10^4 \leq 0, \\ g_3 &\equiv 5 \cdot 10^5 |z_2 - z_1| - 5 \cdot 10^3 \leq 0, \end{aligned} \right\} \quad (2.21)$$

buckling constraints

$$\left. \begin{aligned} g_4 &\equiv -10(z_1 + z_2) - \pi^2 \beta b_1 \leq 0, \\ g_5 &\equiv -10z_2 - \pi^2 \beta b_2 \leq 0, \\ g_6 &\equiv -10(z_2 - z_1) - \pi^2 \beta b_3 \leq 0, \end{aligned} \right\} \quad (2.22)$$

and displacement constraints

$$\left. \begin{aligned} g_7 &\equiv |z_1| - \delta_1 \leq 0, \\ g_8 &\equiv |z_2| - \delta_2 \leq 0. \end{aligned} \right\} \quad (2.23)$$

The state equations, in matrix form, are

$$\frac{\sqrt{2} \cdot 10^6}{4} \begin{pmatrix} b_1 + b_3 & b_1 - b_3 \\ b_1 - b_3 & b_1 + b_3 + 2\sqrt{2}b_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} - \begin{pmatrix} P \cos \alpha \\ P \sin \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.24)$$

Finally, the constraints must be satisfied for any  $\alpha$  in an interval  $[\alpha_1, \alpha_2]$ . Since the state equations are simple, the state variables could be eliminated, explicitly. As a test of the POD method, however, this problem is solved with state equations as given.

The maximum number of local maximum points of  $g_1(\alpha)$  is conjectured to be two. Hence two values of  $\alpha$  are given as estimates. During iterative solution, the nominal local maximum points may approach each other, showing that there is just one maximum point. If this happens, one of the inner problems is removed.

For numerical examples that follow, let  $\beta = 0.07958$  (corresponding to a member of circular cross section),  $P =$

$4 \times 10^4$  lbs, and two load ranges and displacement constraints be considered, as follows: (i)  $-45^\circ \leq \alpha \leq 90^\circ$ ,  $\delta_1 = \delta_2 = 0.005"$ , and (ii)  $0^\circ \leq \alpha \leq 90^\circ$ ,  $\delta_1 = 0.005"$ ,  $\delta_2 = 0.004"$ .

### 2.3.3 Vibration Isolator Design Problem

As a third example, optimal design of a vibration isolator, shown in Figure 2, is considered. The objective is to minimize the maximum displacement of the main mass by attaching a secondary spring-mass system. This problem is treated analytically in Den Hartog [30], with an infinite range of exciting frequency, and in [31], for an arbitrary range of exciting frequency. In the latter treatment, the problem is solved approximately by enforcing the constraints on a preselected grid of points over the given range of frequencies.

The problem, in POD form, is given as: Minimize

$$b_{n+1} \quad (2.25)$$

subject to

$$g_1 \equiv x_1^2(\xi, f, \alpha) - b_{n+1} \leq 0, \quad (2.26)$$

$$g_2 \equiv \left| \frac{x_2(\xi, f, \alpha) - x_1(\xi, f, \alpha)}{x_1(\xi, f, \alpha)} \right| - Q_{\max} \leq 0, \quad (2.27)$$

where

$$x_1 = \sqrt{\frac{A^2 + B^2}{C^2 + D^2}}, \quad x_2 = \sqrt{\frac{A'^2 + B^2}{C^2 + D^2}},$$

and

$$A = -(\alpha^2 - f^2),$$

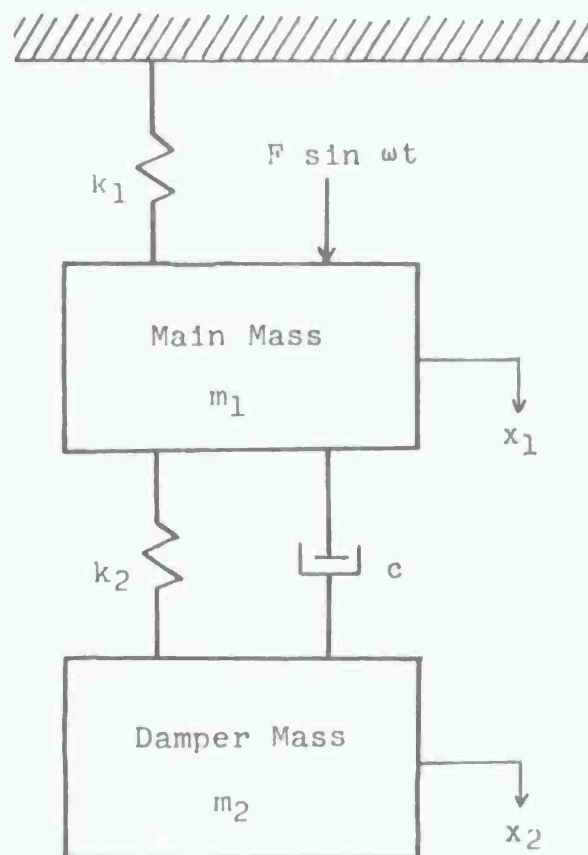


Figure 2 Vibration Isolator

$$A' = f^2,$$

$$B = 2\xi\alpha,$$

$$C = -\mu f^2 \alpha^2 + (\alpha^2 - 1)(\alpha^2 - f^2),$$

$$D = 2\xi\alpha(\alpha^2 - 1 + \mu\alpha^2).$$

Here

$\xi$  = damping ratio ( $c/c_c$ ,  $c_c$ =critical damping),

$f$  = ratio of uncoupled natural frequencies of the absorber and the main mass ( $\omega_n/\Omega_n$ ),

$\alpha$  = ratio of exciting frequency to uncoupled natural frequency of main mass ( $\omega/\Omega_n$ ),

and  $\mu$  = mass ratio of the absorber to main mass ( $m_2/m_1$ ).

The variables  $\xi \equiv b_1$  and  $f \equiv b_2$  are chosen as design parameters and  $\alpha$  is the environmental parameter. The coordinates,  $x_1$  and  $x_2$  represent the extreme displacements of the main mass and absorber, respectively. The value  $Q_{\max}$  is selected as the upper limit for relative motion of the two masses, sometimes called a rattlespace constraint.

Design parameter constraints are

$$g_3 \equiv \begin{cases} -\xi \leq 0, \\ \xi - \xi_{\max} \leq 0, \end{cases} \quad (2.28)$$

$$g_4 \equiv \begin{cases} -f \leq 0, \\ f - f_{\max} \leq 0, \end{cases} \quad (2.29)$$

and free parameter constraints are

$$q_1 \equiv \begin{cases} \alpha_{\min} - \alpha \leq 0, \\ \alpha - \alpha_{\max} \leq 0. \end{cases} \quad (2.30)$$



For subsequent calculation  $\mu=0.1$  and  $Q_{\max}=3.0$ , and two ranges of  $\alpha$  are considered: (i)  $0.5 \leq \alpha \leq 1.5$ , and (ii)  $0.9 \leq \alpha \leq 1.25$ . Since the parametric constraints are not convex in  $\alpha$ , it is important to keep every possible local maximum point. From physical characteristics of a 2 degree-of-freedom system, one may conclude that  $g_1$  has at most two local maximum points over any interval of  $\alpha$ . Hence, two estimates of local maximum points are sufficient. For a given range of  $\alpha$ , there may be only one maximum point at the initial design estimate. Since one may not account for the case in which a new local maximum point appears, it is important to have a reasonable design estimate, or to repeat the solution with different initial design estimate.

### 2.3.4 Optimum Damping in Linear

#### Dynamic Systems [McMunn]

Assuming that synchronous, steady state harmonic forces are the only input, an  $n$  degree-of-freedom dynamical system can be described by the differential equations

$$M\ddot{x} + D\dot{x} + Kx = Pe^{i\omega t} \quad (2.31)$$

where

$M, D, K \equiv$  mass, damping, and stiffness matrices,  
 respectively (the elements of  $D$  are linear  
 combinations of damping coefficients),  
 $x \equiv$  column vector of coordinates,

$P \equiv$  column vector of forces,

$\omega \equiv$  angular frequency.

The design concept chosen here is to minimize the maximum response, over a given range of excitation frequencies. Design variables are damping coefficients of the system. The environmental parameter in this case is the excitation frequency. McMunn's method is restricted to finding saddle points, whereas the present treatment makes no such restriction.

Following McMunn [32], harmonic motion is presumed in the form  $x = X e^{i\omega t}$ . Then the governing equation is

$$T(c, \omega)X = P, \quad (2.32)$$

where

$$T(c, \omega) = -\omega^2 M + i\omega D(c) + K. \quad (2.33)$$

If one excludes systems that have infinite response in the frequency interval,  $T(c, \omega)$  is invertible. By Cramer's rule,

$$\phi(c, \omega) = \left| \frac{X_1}{P_j} \right| = \frac{\tilde{P}_{2(n-1)}(c, \omega)}{\tilde{Q}_{2n}(c, \omega)}, \quad (2.34)$$

where  $\tilde{P}_{2(n-1)}$  and  $\tilde{Q}_{2n}$  are real polynomials in  $\omega^2$ , of degree  $2(n-1)$  and  $2n$ , respectively;  $X_1$  is the complex amplitude of the displacement of the 1-th mass; and  $P_j$  is the amplitude of the input force on the  $j$ -th mass. In practice, the explicit form of  $\phi(c, \omega)$  is difficult to attain. Hence, a set of state equations is retained

in the POD formulation, as follows: Let real and imaginary parts be separated such that

$$T(c, \omega) = T_1(c, \omega) + i T_2(c, \omega), \quad (2.35)$$

and

$$X = u - i v, \quad (2.36)$$

then the state equations are, with the state variable

$$z \equiv \{u, v\},$$

$$\begin{pmatrix} T_1 & T_2 \\ T_2 & -T_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix}, \quad (2.37)$$

and the objective function is given, in terms of the state variable  $z$ , as

$$\phi(z) = [u_1^2 + v_1^2]/P_j^2. \quad (2.38)$$

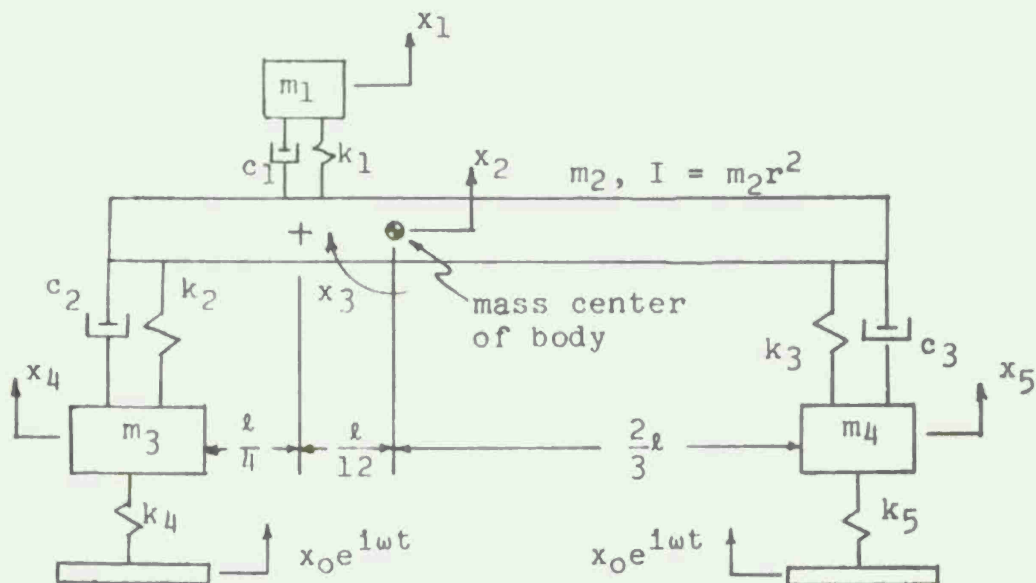
It is noted that  $\phi(c, \omega) \geq 0$  and  $\lim_{\omega \rightarrow \infty} \phi(c, \omega) \rightarrow 0$ . Hence, the case of an infinite range of frequencies can be safely replaced by a finite interval of frequencies such that the  $\omega$ -constraint set  $A$  is closed and bounded.

The problem of optimum damping may now be formulated as: Minimize

$$\max_{\omega} \phi(z) \quad (2.39)$$

subject to the state equations, Eqs. (2.37) and the design variable constraints  $c_1 \geq 0$ . To fit the POD formulation, an artificial design variable  $b_{n+1}$  is introduced, as described in Section 2.2.1.

The following numerical example is taken from McMunn



$$\mu_1 = \frac{m_1}{m_2} = 0.250$$

$$\Omega_2^2 = \left( \frac{k_2 + k_4}{m_2} \right) / \omega_{10}^2 = 2.18$$

$$\mu_2 = \frac{m_3}{m_2} = \frac{m_4}{m_2} = 0.150$$

$$\Omega_3^2 = \left( \frac{k_3 + k_5}{m_3} \right) / \omega_{10}^2 = 80.8$$

$$\beta = \frac{l}{r} = 13.40$$

$$\zeta_1 = c_1 / (2\sqrt{k_1 m_1}) \equiv b_1$$

$$\Omega_1^2 = (k_1 / m_1) / \omega_{10}^2 = 1.64$$

$$\zeta_2 = c_2 / (2\sqrt{(k_2 + k_4) m_3}) \equiv b_2$$

$$\alpha = \omega / \omega_{10}$$

$$\zeta_3 = c_3 / (2\sqrt{(k_3 + k_5) m_3}) \equiv b_3$$

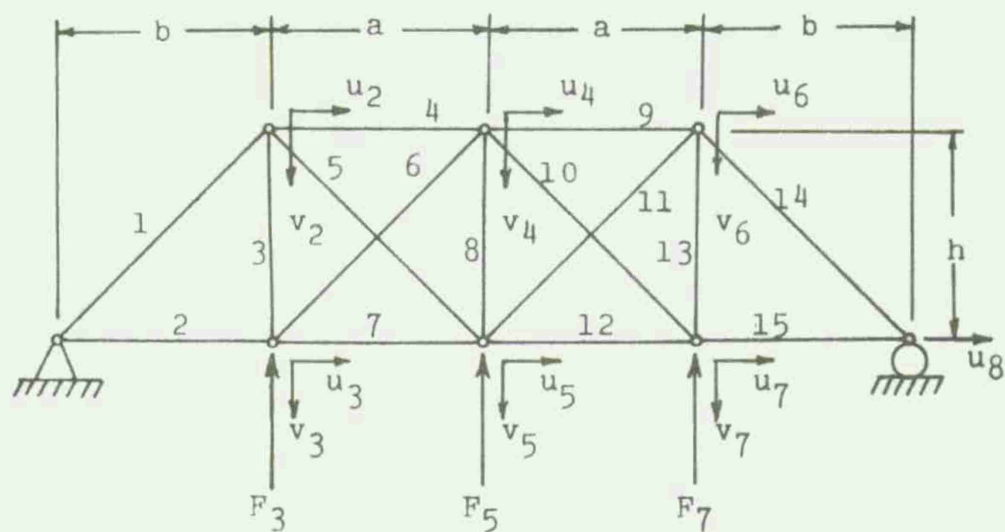
Figure 3 A Five Degree-of-Freedom  
Vehicle Model

[32]: The system to be designed, Figure 3, is a five degree-of-freedom model of a vehicle traveling on a sinusoidal road. The dimensions of the vehicle and the magnitudes of masses and spring constants are the same as in [32], and shown in Figure 3. The parameters  $m_1$ ,  $c_1$ , and  $k_1$  depict the driver and seat attached to the main body, denoted by mass  $m_2$  and a moment of inertia  $I=m_2r^2$ . The parameters  $m_3$  and  $m_4$  are the masses of the wheels, axles, and associated mechanism of the front and the rear of the vehicle, while  $k_4$  and  $k_5$  represent tire stiffnesses. In the present example, the system is excited by a synchronous harmonic displacement of the base, and the cost function is taken as the square of the ratio of the amplitudes of  $m_1$  and the base. The dynamic equations may be written in terms of dimensionless parameters, as given in [32]. Numerically reduced equations, after substituting values of the parameters, are given in Appendix A.

### 2.3.5 Bridge Design Problem

As a more realistic test of the POD formulation, design of a bridge is considered, under load by a vehicle passing from right to left. The bridge is composed of a truss structure over a beam element. For structural analysis, the truss structure is modeled by 15 elements, as shown in Figure 4a. The beam element, to be supported by the truss, is shown in Figure 4b. The vehicle is

## a. Truss Structure



## b. Beam Element

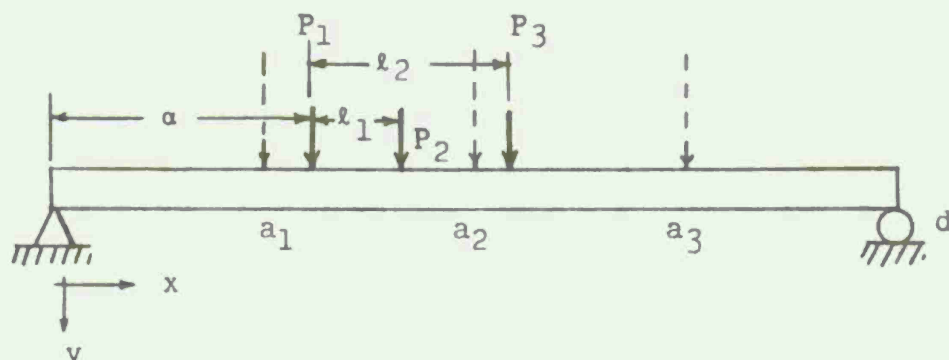


Figure 4 A Bridge Structure



represented by three point loads applied to the beam, at a distance  $\alpha$  from the left end. The design objective is to determine the cross sectional area of each truss member and the stiffness of the beam element to minimize weight of the structure, under certain constraints. For simplicity, only the stress and displacement constraints will be considered, assuming static loading. The environmental parameter is the location of the vehicle on the bridge,  $\alpha$ . The state equations for the system are derived in Appendix B, where the state variables are composed of generalized displacements and the generalized forces acting at the interacting boundary between the truss structure and the beam element. As is usually the case with structural design problems, the state equations are linear in the state variables, with the form,

$$K(b)z = F(\alpha),$$

where the matrix  $K$  is a function of the design variables only. The linear nature of the equations yields computational savings, since it is possible to solve the system of equations for the parametric constraints, where  $\alpha$  assumes different values, only once for each outer iteration. If the dimension of  $z$  is small, an explicit inversion of  $K$  gives less programming effort. For other problems, such as the one given in Section 2.3.4, the coefficient matrix is a function of the free parameter  $\omega$ ,



as well as the design variable  $c$ .

Stress constraints for the truss members are represented by

$$\sigma_c \leq \sigma \leq \sigma_T \quad (2.40)$$

where the axial stress  $\sigma$  is obtained as explained in Appendix B and  $\sigma_c$  and  $\sigma_T$  are the yield stresses for compression and tension. For the beam element, a moment constraint of the following type is considered;

$$M(x) \leq \frac{M_Y}{SF} \quad (2.41)$$

where  $M(x)$  is the bending moment along the beam, and  $M_Y = \sigma_Y S$  is the yield moment. The safety factor  $SF$  is chosen as 1.85 and the section modulus  $S$  is chosen as  $S = 0.58 \cdot I^{3/4}$ .

The objective function assumes the form,

$$J = \sum_{i=1}^{15} \rho_1 A_1 L_1 + \rho_0 A_0 d, \quad (2.42)$$

where  $\rho_1$ ,  $L_1$ , and  $A_1$  are the specific weight, the length, and the cross sectional area of  $i$ -th member of the truss, respectively. The parameters  $\rho_0$ ,  $d$ , and  $A_0$  are the specific weight, length, and cross sectional area of the beam. In the computation,  $A_0 = 0.58 \cdot I^{1/2}$  is used and  $A_1$ ,  $A_2, \dots, A_8$ , and the moment of inertia  $I$  are chosen as design variables, assuming a geometrical symmetry of the structure.

## 2.4 Mathematical Preliminaries

In the field of nonlinear programming, a vast amount of literature is available. The following texts are cited as general references: [12,25,33-37]. For future reference, some concepts and definitions needed for development of a POD algorithm are summarized here.

### 2.4.1 General Philosophy of Iterative Methods

It has been observed that an analytical solution to an optimization problem is practically impossible. Most of the literature on computational techniques use iterative methods and obtain a local solution to the optimization problem. The basic philosophy of any iterative method is to generate a sequence of nominal solutions from an estimated point, such that the sequence converges to a limit point or a desirable point that satisfies certain conditions. Two basically different points of view are possible, depending on how the designer improves his nominal design. The first point of view is indirect. It seeks a solution of necessary conditions that must be satisfied at a local solution point. The second point of view is direct, in the sense that the designer tries to improve the design, using the local nature of the problem around the nominal point. Most of the iterative methods for solving a nonlinear programming problem are a combination of the two. In the second point of view, an

approximate problem is generated from the original problem, through function approximations, usually with a tangent hyperplane at the nominal point. The approximate problem is then solved using necessary conditions.

#### 2.4.2 Definitions and Theorems for NLP Problems

Important definitions and theorems are described for the following nonlinear programming problem (NLP):

Minimize

$$f(x) \tag{2.43}$$

subject to constraints

$$g(x) \leq 0, \tag{2.44}$$

and

$$h(x) = 0, \tag{2.45}$$

where  $x \in \mathbb{R}^n$ ,  $g$  is an  $l$ -dimensional vector,  $h$  is an  $m$ -dimensional vector, and  $m \leq n$ .

A point  $x \in \mathbb{R}^n$  that satisfies all the functional constraints is said to be feasible. The set of all such points is called the constraint set and will be denoted by  $D$ . The following definitions are often referred to in later chapters. An inequality constraint  $g_1(x) \leq 0$  is said to be active at a point  $x$  if  $g_1(x) = 0$  and inactive at  $x$  if  $g_1(x) < 0$ . By convention, each equality constraint is considered active at any feasible point. This definition is of rather theoretical interest, since in numerical analysis one rarely satisfies the equality. Hence, the

following concepts are more useful for numerical development. Given a point  $x \in R^n$  and a real number  $\epsilon \geq 0$ , the inequality constraint is said to be  $\epsilon$ -active at  $x$  if  $g_1(x) \geq -\epsilon$ , and inactive otherwise. All equality constraints will be considered  $\epsilon$ -active. An  $\epsilon$ -active index set  $I_\epsilon$  at  $x$  is defined as  $I_\epsilon(x) = \{i | g_i \text{ is } \epsilon\text{-active}\}$ . Here, intentionally, the indices of the equality constraints are not included in the set  $I_\epsilon$ . The usefulness of this definition is illustrated in Chapter 6. The index set  $I$  is used to denote  $I_\epsilon$  when  $\epsilon=0$ .

The existence of a solution to NLP is guaranteed by the following theorem [33,34]: If  $f(x)$  is continuous on  $D$  and  $D$  is closed and bounded in  $R^n$  (compact for a more general space), then,  $f(x)$  has a minimum (and a maximum) in  $D$ .

For NLP without equality constraints, a vector  $d$  is said to be a feasible direction at  $x \in D$  if there is a  $\bar{\beta} > 0$  such that  $x + \beta d \in D$  for all  $\beta$ ,  $0 \leq \beta \leq \bar{\beta}$ . The set of directions  $r_x = \{d | d \in R^n, x + \beta d \in D \text{ for some } \beta\}$  is called the set of feasible directions for  $x \in D$  and is a cone. The set  $r_x$  is a convex cone if the set  $D$  is convex. For a definition of convexity and cone, see Appendix C. For a set of linear equality constraints, this definition could still be applied. For more general equality constraints, the concept of tangent direction is introduced [38].

In the following, the functions  $f$ ,  $g$ , and  $h$  are assumed once differentiable. At a feasible point  $x_0$ , the first order constraint qualification is said to hold if every feasible direction at  $x_0$  is tangent to a once differentiable arc (a directed curve) emanating from  $x_0$  and contained in the constraint set. Roughly, the first order constraint qualification, or regularity condition, is said to be satisfied if a small move along any feasible direction must leave the point still in the constraint set. To identify the class of constraints that satisfy this condition at  $x_0$ , the following theorem is cited; the first order constraint qualification holds at a feasible point  $x_0$  if the gradient vectors of the active constraints are linearly independent. Such a point will be called a regular point of the constraint set. At a regular point  $x_0$ , a tangent subspace is defined by the set  $M = \{y \mid \{\nabla g_i(x_0), i \in I, \forall h(x_0)\} \cdot y = 0\}$ , where  $y$  is an  $|I| + m \leq n$  dimensional vector.  $|I|$  is the number of elements in the set  $I$ .

It is obvious that if at a point there is a direction that does not lead out of the constraint set and along which the objective function decreases, then such a point can not be a minimum point. Based on this observation, a necessary condition for a regular point  $x_0$  to be a local minimum point is stated: The gradient  $\nabla f$  of the objective function is orthogonal to the tangent plane. That is,

$\nabla f$  is a linear combination of the gradients of active constraints. Stated differently, the projection of the gradient  $\nabla f$  onto the tangent plane is zero. More precisely, the following Kuhn-Tucker Theorem is stated [33]: If  $x_0$  is a relative (local) minimum point for NLP and is a regular point for the constraints, then there exists a vector  $\lambda \in \mathbb{R}^m$  and a vector  $\mu \in \mathbb{R}^l$ , with  $\mu \geq 0$ , such that

$$\nabla L(x_0, \lambda, \mu) = 0, \quad (2.46)$$

and

$$\mu_i g_i(x_0) = 0, \quad (2.47)$$

where

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x) \quad (2.48)$$

is called the Lagrangian. The vectors  $\mu$  and  $\lambda$  are called Lagrange multipliers, corresponding to the inequality constraints and equality constraints, respectively.

According to the Kuhn-Tucker theorem, the Lagrangian functional is stationary at  $x_0$ . If the regularity assumption on  $x_0$  is omitted, one can only conclude that the functional  $\lambda_0 f(x) + \lambda^T h(x) + \mu^T g(x)$  is stationary. If  $x_0$  is regular,  $\lambda_0$  may be chosen as one and the optimization problem is said to be normal [37]. If  $x_0$  is not a regular point, then  $\lambda_0 = 0$  and the problem is abnormal.

Assuming that functions in NLP are twice differentiable, one can show that the Hessian matrix of second partial derivatives of the Lagrangian with respect to  $x$  is positive semidefinite on the tangent subspace of the



active constraints at  $x_0$  [34]. This is called second order necessary conditions. If the Hessian matrix is positive definite, then, this strengthened condition, combined with Kuhn-Tucker conditions, constitutes a second order sufficient condition for  $x_0$  to be a strict relative minimum.

#### 2.4.3 Approximations of the Problem NLP

As stated at the beginning of this section, an iterative method is established by first forming an approximate problem, using Taylor series expansions. If second and higher order terms are deleted, the approximation is linear. If third or higher order terms are deleted, one has a second order approximation. The most common approximation around  $x$  is first order,

$$f(y) - f(x) \approx \frac{df(x)}{dx} (y - x), \quad (2.49)$$

where  $\|y-x\|$  is small enough that the higher order terms are negligibly small. The left hand side of Eq. (2.49) is denoted  $\delta f(x)$  and  $(y-x)$  is denoted  $\delta x$ . These quantities are called variations. Hence,

$$\delta f(x) = \frac{df(x)}{dx} \delta x + O(\delta x^2), \quad (2.50)$$

where  $O(x)/|x| \leq K$  for small  $x$  for some constant  $K$ .

Similarly, a variation of  $f$  up to second order is expressed by

$$\delta f(x) = \frac{df(x)}{dx} \delta x + \frac{1}{2} \delta x^T \frac{d^2 f(x)}{dx^2} \delta x + O(\delta x^3), \quad (2.51)$$



where  $\delta x$  has same meaning of a finite variation. In the following studies, the remainder terms for the expressions of variation of  $f$  will be omitted, with the understanding that  $||\delta x||$  is small enough so that the expansion has negligible higher order terms.

#### 2.4.4 Gradient Projection with Constraint Error Compensation

The first order expansion is described here to solve a NLP. The method is called a "gradient projection method with constraint error compensation" [33], since the direction of movement is defined as the vector sum of a projection of the negative gradient onto a subspace and another vector that compensates any constraint error at the current point. The following result is derived from the Kuhn-Tucker necessary condition, applied to the first order approximate problem. The approximate problem is of the form: Minimize

$$\delta f = l^J{}^T \delta b \quad (2.52)$$

subject to

$$\delta g = l^\phi{}^T \delta b \leq \Delta g, \quad (2.53)$$

where  $l^J$  and  $l^\phi$  are gradients of  $f$  and  $g$  in the design space, i.e., when equality constraints or state equations are involved, the state variables are locally eliminated. The particular form of these gradients is derived for each problem in the following chapters. The quantity  $\Delta g$  is the

desired reduction in the current constraint violation.

The solution of the above approximate problem can result in a large  $\delta b$ , so the linear approximation would no longer be valid. Hence, a stepsize restriction is imposed on  $\delta b$ ; namely:

$$\delta b^T W \delta b \leq \eta^2, \quad (2.54)$$

where  $\eta$  is a small constant and  $W$  is a positive definite weighting matrix. The choice of  $W$  and  $\eta$  is based on computational skill, suggestions for which are given in Section 6.3.2. Also, since this iterative method is local in nature, around the current nominal design, the inequality constraints that are inactive are not of interest. Therefore, the matrix of gradients  $\ell^\phi$  contains only columns associated with active constraints. With this formulation, the solution of the Kuhn-Tucker necessary conditions for the local problem is [33]:

$$\delta b = - \frac{W^{-1}}{2\gamma} (\ell^J - \ell^\phi \lambda), \quad (2.55)$$

$$\lambda = -M_{\phi\phi}^{-1} (\ell^\phi{}^T W^{-1} \ell^J + 2\gamma \Delta g), \quad (2.56)$$

where

$$M_{\phi\phi} = \ell^\phi{}^T W^{-1} \ell^\phi, \quad (2.57)$$

and  $\lambda$  and  $\gamma$  are multipliers. A choice of  $\gamma$ , in relation to  $\eta$ , is suggested in Section 6.3.2.

Instead of the direct inversion of matrix  $M_{\phi\phi}$ , the following scheme is used in computation. Let

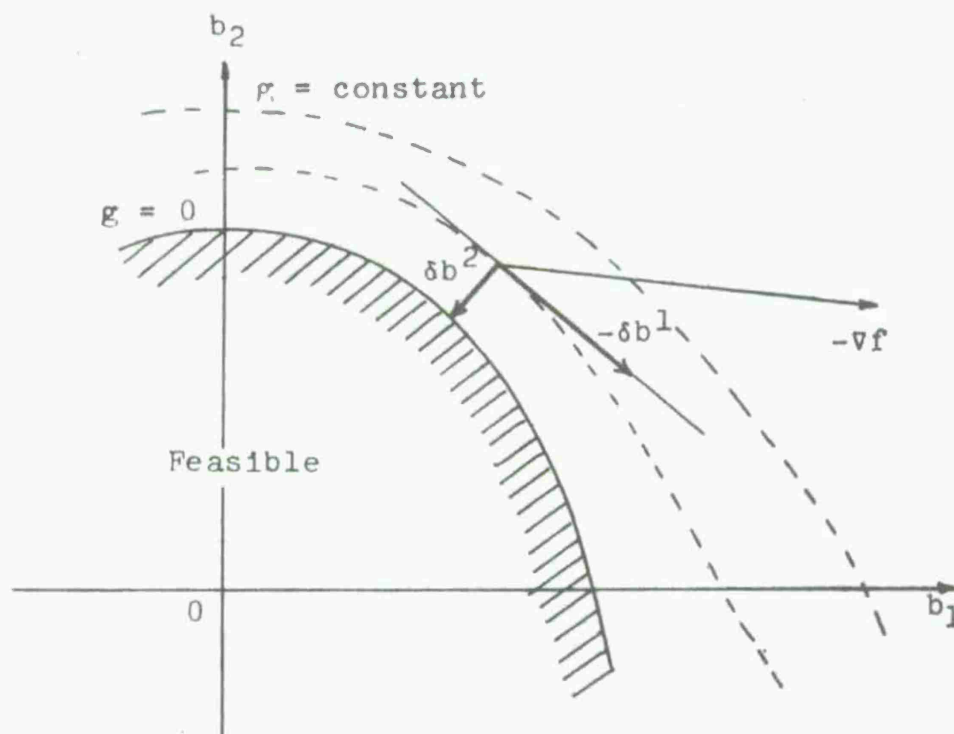


Figure 5 Gradient Projection Method  
with Constraint Error Compensation

$$\lambda = \lambda^1 - 2\gamma\lambda^2, \quad (2.58)$$

where

$$M_{\phi\phi}\lambda^1 = -\ell\phi^T W^{-1}\ell^J, \quad (2.59)$$

and

$$M_{\phi\phi}\lambda^2 = \Delta g. \quad (2.60)$$

Then, one has

$$\delta b = -\frac{W^{-1}}{2\gamma}(\ell^J + \ell\phi\lambda^1) + W^{-1}\ell\phi\lambda^2. \quad (2.61)$$

Putting

$$\delta b = -\frac{\delta b^1}{2\gamma} + \delta b^2, \quad (2.62)$$

where

$$\delta b^1 = W^{-1}(\ell^J + \ell\phi\lambda^1) \quad (2.63)$$

$$\delta b^2 = W^{-1}\ell\phi\lambda^2, \quad (2.64)$$

one can verify the relations [39,40]:

$$(1) \quad \delta b^2 W \delta b^1 = 0, \quad (2.65)$$

$$(2) \quad \ell\phi^T \delta b^1 = 0, \quad (2.66)$$

$$(3) \quad \ell\phi^T \delta b^2 = \Delta g, \quad (2.67)$$

$$(4) \quad -\ell^J W \delta b^1 \leq 0. \quad (2.68)$$

The geometrical situation is shown in Figure 5, in a two dimensional design space with one active constraint.

#### 2.4.5 Analytical Properties of Max-Value Functions

Recently, a number of authors [5,6,41] have studied functions of the type  $\mu(x) = \max_{\alpha} \phi(x, \alpha)$  and have applied their results to solution of practical problems such as mathematical programming with continuum of constraints

and the theory of best approximation. The distinguishing feature is that the function  $\mu(x)$  is not differentiable, in general, but is only quasi-differentiable, under mild assumptions. In the remainder of this section, relevant properties of the function  $\mu(x)$  are summarized, mainly from Pshenichnyi [6]. For definitions of basic terminologies in functional analysis, see Appendix C.

For a bounded convex functional  $\phi(x)$ ,  $x$  in a Banach space  $B$ , i.e., a convex functional that is bounded above on every bounded region  $D$ , the set of support functionals  $M(x_0)$  to  $\phi(x)$ , at  $x_0$ , is defined by

$$M(x_0) = \{x^* \mid x^* \in B^*, \phi(x) - \phi(x_0) \leq x^*(x - x_0) \text{ for all } x \in B\}, \quad (2.69)$$

where  $B^*$  is the normed dual of  $B$ . It is shown in [6] that  $M(x_0)$  is convex, weak\* closed, and bounded. The term "sub-differentials" of  $\phi(x)$  at  $x_0$  has been used for the set of support functionals by Rockafellar [41]. If  $\phi(x)$  is Gateaux differentiable at  $x_0$ ,  $M(x_0)$  is simply the single functional  $x_0^*$ , the Gateaux differential of  $\phi(x)$  at  $x_0$ . The directional differential, defined by

$$\frac{\partial \mu}{\partial e} = \lim_{\lambda \rightarrow +0} \frac{\phi(x_0 + \lambda e) - \phi(x_0)}{\lambda}, \quad (2.70)$$

is given by the formula [6]

$$\frac{\partial \mu}{\partial e} = \max_{x^* \in M(x_0)} x^*(e) \quad (2.71)$$

for all  $x_0$  and  $e$ . Motivated by this result, a broader class of functionals is defined. A functional is said

to be quasi-differentiable at  $x_0$  if there exists a convex, weak\* closed set  $M(x_0)$  such that Eq. (2.71) holds. The class of Gateaux differentiable functionals and bounded convex functionals are in the class of quasi-differentiable functionals.

Now consider the function

$$\mu(x) = \max_{\alpha \in A} \phi(x, \alpha) \quad (2.72)$$

where  $A$  is a closed and bounded subset in a Euclidean space. For convenience, define the "answering set" [5] at  $x$ ,

$$A(x) = \{\alpha \mid \phi(x, \alpha) = \mu(x)\}. \quad (2.73)$$

If  $\phi(x, \alpha)$  is a functional that is continuous in  $x$  and  $\alpha$ , where  $x$  is in a Banach space  $B$  and  $\alpha \in A$ , then  $\mu(x)$  is continuous. Moreover, let

$$\phi(x_0 + \lambda e, \alpha) = \phi(x_0, \alpha) + \lambda \frac{\partial \phi(x_0, \alpha)}{\partial e} + \lambda \gamma(\lambda, \alpha),$$

for  $\lambda > 0$ , where  $\gamma(\lambda, \alpha) \rightarrow 0$  uniformly in  $\alpha$  as  $\lambda \rightarrow +0$ .

Then,  $\mu(x)$  is differentiable in the direction  $e$  and

$$\frac{\partial \mu(x_0)}{\partial e} = \sup_{\alpha \in A(x_0)} \frac{\partial \phi(x_0, \alpha)}{\partial e}. \quad (2.74)$$

Furthermore, if  $\phi(x, \alpha)$  is quasi-differentiable at  $x_0$ , for every  $\alpha$ , then  $\mu(x)$  is quasi-differentiable at  $x_0$ , with

$$M(x_0) = \overline{\text{co}} \bigcup_{\alpha \in A(x_0)} M(x_0, \alpha), \quad (2.75)$$

where  $M(x_0, \alpha)$  is the set of support functionals to  $\phi(x, \alpha)$  at  $x_0$ ,  $M(x_0)$  is the set of support functionals to  $\mu(x)$



at  $x_0$ , and  $\overline{\text{co}} K$  denotes the weak\* closure of the convex hull of  $K$  [6].

Existence of the directional derivative and its properties, such as the continuity of the directional derivative when  $A(x)$  is composed of a single point, are discussed in Danskin [5]. For future reference, an important theorem is cited: If  $\phi(x, \alpha)$  is continuous in  $x$  and  $\alpha$ , and if  $\frac{\partial \phi(x, \alpha)}{\partial x}$  is continuous with respect to  $x$ , then  $\mu(x)$  is differentiable in any direction  $e$  at any point  $x$  and

$$\frac{\partial \mu(x)}{\partial e} = \max_{\alpha \in A(x)} \frac{\partial \phi(x, \alpha)}{\partial e}. \quad (2.76)$$

If  $A(x)$  consists of one point  $\alpha(x)$  only, then  $\mu(x)$  is continuously differentiable,

$$\frac{\partial \mu(x)}{\partial x} = \frac{\partial \phi(x, \alpha(x))}{\partial x} \quad (2.77)$$

and  $\alpha(x)$  is continuously dependent on  $x$  [5,6].

#### 2.4.6 Theory of Parametric Nonlinear Programming

Consider the problem of minimizing a continuously differentiable function  $f(x)$  subject to the constraint

$$\mu(x) \equiv \max_{\alpha \in A} g(x, \alpha) \leq 0, \quad (2.78)$$

where  $g(x, \alpha)$  is continuous with respect to  $\alpha$  and  $x$  and has a continuous gradient  $\frac{\partial g(x, \alpha)}{\partial x}$ ,  $x \in \mathbb{R}^n$ . According to the theorem cited above,  $\mu(x)$  is quasi-differentiable with the set of support functionals given by



$$M(x_0) = \text{co} \left( \bigcup_{\alpha \in A(x_0)} \frac{\partial g(x_0, \alpha)}{\partial x} \right). \quad (2.79)$$

It can be shown [6] that the necessary condition for  $x_0$  to be the minimum point is that there exist nonnegative constants  $\lambda_0$ , and  $\lambda$ , not both zero, such that

$$\lambda_0 \frac{\partial f(x_0)}{\partial x} + \lambda c = 0 \quad (2.80)$$

and

$$\lambda \mu(x_0) = 0,$$

where  $c \in M(x_0)$ . From Eq. (2.79), and Caratheodory's theorem [41], the vector  $c$  can be represented in the form,

$$c = \sum_{i=1}^{n+1} \lambda_i \frac{\partial g(x_0, \alpha_i)}{\partial x}, \quad (2.82)$$

where

$$\alpha_i \in A(x_0), \quad \lambda_i \geq 0, \quad \sum_{i=1}^{n+1} \lambda_i = 1.$$

Hence a necessary condition for  $x_0$  to be a solution to the above problem is that there exist constants  $\lambda_0 \geq 0$  and  $\gamma_i \geq 0$ , not all zero, and points  $\alpha_i \in A(x_0)$ ,  $i=1, \dots, n+1$ , such that

$$\lambda_0 \frac{\partial f(x_0)}{\partial x} + \sum_{i=1}^{n+1} \gamma_i \frac{\partial g(x_0, \alpha_i)}{\partial x} = 0. \quad (2.83)$$

Furthermore, if  $f(x)$  and  $g(x, \alpha)$  are convex functionals for each  $\alpha$  and if  $\mu(x_1) < 0$  for some  $x_1$  (interior point condition), then it is necessary and sufficient that there exist  $\alpha_i \in A(x_0)$ ,  $i=1, \dots, n+1$ , such that  $x_0$  is a solution to the following problem

$$\min f(x) \quad (2.84)$$

subject to

$$g(x, \alpha_i) \leq 0, \quad i=1, \dots, n+1. \quad (2.85)$$

This result is very similar to the Fritz John Theorem [1], which states that at most  $n$   $\alpha_i$ 's are sufficient to characterize the solution of the original continuum problem by a discrete problem with at most  $n$  equality constraints. If the above result is applied to the Chebyshev approximation problem discussed in Section 2.2, the fundamental theorem of Chebyshev approximation is obtained [6].

While the justification of the assumptions is often difficult, certain continuity assumptions can safely be made in practice. For the following development of numerical techniques, let  $g(x, \alpha)$  be continuous in  $x$  and  $\alpha$ , where  $x \in R^n$  and  $\alpha \in R^p$ , and let  $\frac{\partial g(x, \alpha)}{\partial x}$  be continuous in  $x$ . Then, from the above,  $v(x) = \max_{\alpha \in A} g(x, \alpha)$  is directionally differentiable, where  $A$  is a closed bounded set. Also, there exist points  $\alpha_i, i=1, \dots, n+1$ , in  $A(x_0)$  that characterize the original problem. The difficulty with the problem at hand is that the points  $\alpha_i, i=1, \dots, n+1$  are not known beforehand and the numerical procedure that follows is very much dependent on how these points are obtained. As observed in practical problems, it will be assumed that the answering set  $A(x)$ , for each  $x$ , consists of a finite number of isolated points,  $\alpha_i, i=1, \dots, s$ , such that it is always possible to have a closed neighborhood

$U_1$ , corresponding to  $\alpha_1$  in  $A(x)$ , disjoint from each other. Thus, the expansion procedure discussed in Section 2.1 is realizable. The expanded form of the problem is,

$$\min f(x) \quad (2.86)$$

subject to

$$\mu_1(x) \equiv \max_{\alpha \in U_1} \phi(x, \alpha) \leq 0, \quad i=1, \dots, s, \quad (2.87)$$

where  $U_1 \subset A$ .

In this case, it is evident that the answering set  $U_1(x)$  for  $\mu_1(x)$  is composed of a single point  $\alpha_1(x)$ , where  $\alpha_1(x)$  is continuous in  $x$  and  $\mu_1(x)$  is in the class of continuously differentiable functions, making it possible to apply the theory of nonlinear programming. These features are utilized implicitly in the numerical algorithms that are developed more formally in the following chapters.

## CHAPTER 3

### A FIRST ORDER ALGORITHM FOR THE POD PROBLEM

#### 3.1 Introduction

Iterative algorithms for design improvement can be based on varying degrees of precision of approximation of the problem. In this chapter, first order approximations will be used and a "first order" algorithm will be developed. The basic idea in the development of the algorithm is to determine what constraints the inner problems place on allowable variations of the design variable. Once this information is available, one can apply steepest descent or other iterative design techniques. In the following, an algorithm based on the gradient projection method described in Section 2.4, is given.

#### 3.2 A First Order Algorithm by Gradient Projection

Assuming that the conditions given in Section 2.1.3 are satisfied, the procedure will be as follows:

(1) Estimate  $b^{(1)}$ , solve the inner problems and denote the solutions by the vectors  $\bar{a}_g$  and  $\bar{z}_g$ . When solutions of an inner problem corresponding to the same parametric constraint approach each other, one inner problem must be removed. Thus a "merge" check for the local

maximum points is necessary.

(2) Check the constraints  $g_1(\bar{z}_1, b^{(0)}, \bar{\alpha}_1) \leq 0$ .

(3) Define the index set  $I_\epsilon \equiv \{i | g_i \geq -\epsilon\}$ .

(4) Consider  $g_1, i \in I_\epsilon$ . For perturbations  $\delta b$  around  $b^{(0)}$ , perturbations of  $\bar{\alpha}$  and  $\bar{z}$  must satisfy,

$$\max_{\delta \alpha} \left[ \frac{\partial g_1}{\partial z} \delta z + \frac{\partial g_1}{\partial b} \delta b + \frac{\partial g_1}{\partial \alpha} \delta \alpha \right] \leq \Delta g_1 \quad (3.1)$$

subject to

$$\delta h = \frac{\partial h}{\partial z} \delta z + \frac{\partial h}{\partial b} \delta b + \frac{\partial h}{\partial \alpha} \delta \alpha = 0. \quad (3.2)$$

and

$$\delta q = \frac{\partial q}{\partial \alpha} \delta \alpha \leq 0, \quad (3.3)$$

where derivatives are calculated at  $\bar{\alpha}_1, \bar{z}_1$ , and  $b^{(0)}$ . The quantity  $\Delta g_1$  is the desired reduction in constraint violation, usually taken as  $\Delta g_1 = -g_1(\bar{z}, b^{(0)}, \bar{\alpha})$ . This linearized subproblem may be viewed as a restriction on the variation  $\delta b$ , which must be made more explicit for calculation of solutions.

Since  $\bar{\alpha}$  and  $\bar{z}$  solve the inner problem of Eqs. (2.2) and (2.3), with  $b^{(0)}$  fixed, the Kuhn-Tucker necessary conditions (See Section 2.4) must be satisfied:

$$\frac{\partial L_1}{\partial \alpha} = 0 = -\frac{\partial g_1}{\partial \alpha} + \kappa^T \frac{\partial h}{\partial \alpha} + \mu^T \frac{\partial q}{\partial \alpha}, \quad (3.4)$$

$$\frac{\partial L_1}{\partial z} = 0 = -\frac{\partial g_1}{\partial z} + \kappa^T \frac{\partial h}{\partial z}, \quad (3.5)$$

$$\mu_i q_i = 0, \quad i=1, \dots, r, \quad (3.6)$$

$$\mu_i \geq 0, \quad i=1, \dots, r \quad (3.7)$$

$$h(z, b, \alpha) = 0, \quad (3.9)$$

where

$$L_1 = -g_1 + \kappa^T h + \mu^T q. \quad (3.9)$$

Hence,

$$\frac{\partial g_1}{\partial \alpha} \delta \alpha = (\kappa^T \frac{\partial h}{\partial \alpha} + \mu^T \frac{\partial q}{\partial \alpha}) \delta \alpha \quad (3.10)$$

and

$$\frac{\partial g_1}{\partial z} \delta z = \kappa^T \frac{\partial h}{\partial z} \delta z = -\kappa^T (\frac{\partial h}{\partial b} \delta b + \frac{\partial h}{\partial \alpha} \delta \alpha). \quad (3.11)$$

Therefore, the expression in brackets in Eq. (3.1) is simplified to:

$$\delta g_1 = (\frac{\partial g_1}{\partial b} - \kappa^T \frac{\partial h}{\partial b}) \delta b + \mu^T \frac{\partial q}{\partial \alpha} \delta \alpha, \quad (3.12)$$

so that explicit dependence on  $\delta z$  has been eliminated.

The linearized POD problem is now: Minimize

$$\delta f = \frac{\partial f}{\partial b} \delta b \quad (3.13)$$

subject to

$$\max_{\delta \alpha} [\mu^T \frac{\partial q}{\partial \alpha} \delta \alpha] \leq -(\frac{\partial g_1}{\partial b} - \kappa^T \frac{\partial h}{\partial b}) \delta b + \Delta g_1, \quad (3.14)$$

$$\frac{\partial q}{\partial \alpha} \delta \alpha \leq 0,$$

$$\mu \geq 0.$$

The two inequalities above require that  $\mu^T \frac{\partial q}{\partial \alpha} \delta \alpha \leq 0$ , so  $\max_{\delta \alpha} \mu^T \frac{\partial q}{\partial \alpha} \delta \alpha \leq 0$ . However, the maximum value of zero can be achieved by  $\delta \alpha = 0$ . This observation eliminates explicit dependence in the linearized problem on  $\delta \alpha$ , so the constraint on the design variable is written only in terms of the design variable. The linearized problem of Eqs. (3.13) and (3.14) now becomes simply: Minimize



$$\delta f = \frac{\partial f}{\partial b} \delta b \quad (3.15)$$

subject to

$$\left( \frac{\partial g_1}{\partial b} - \kappa^T \frac{\partial h}{\partial b} \right) \Big|_{\bar{a}_1, \bar{z}_1} \delta b \leq \Delta g_1, \quad 1 \in I_\epsilon, \quad (3.16)$$

where  $\kappa^T$  is given from Eq. (3.5).

With  $\delta b$  solved from this approximate problem, an improved design is  $b^{(1)} = b^{(0)} + \delta b$ . One may now check for convergence and terminate or return to Step (2), with  $b^{(0)}$  replaced by  $b^{(1)}$ .

The procedure for solving the POD problem may now be divided into two independent steps: (1) Solution of sub-problems, and (2) solution of the outer problem given by Eqs. (3.15), and (3.16). Several different versions of the solution procedure are possible, depending on the methods used to implement these two steps. Possible approaches will be discussed in Section 5.2. For Step (2), the solution by gradient projection is exclusively used in the algorithm.

The approximate problem obtained above is exactly the same as is given by Eqs. (2.52) and (2.53), and can be readily solved for a design improvement by imposing a step-size restriction on  $\delta b$ , as in Eq. (2.54);

$$\delta b^T W_b \delta b \leq \eta^2 \quad (3.17)$$

where  $W_b$  is a positive definite weighting matrix and  $\eta$  is a small constant. Following Section 2.4, the solution of the approximate problem is obtained as Eqs. (2.62), (2.63),



and (2.64); where

$$L^J{}^T = \frac{\partial f}{\partial b}, \quad (3.18)$$

$$L^\phi{}^T = \left\{ -\frac{\partial g_1}{\partial b} - \kappa^T \frac{\partial h}{\partial b}, 1 \in I_\epsilon \right\}, \quad (3.19)$$

$$\Delta g = \{\Delta g_1, 1 \in I_\epsilon\}. \quad (3.20)$$

The first order analysis described here leads to a simple computational scheme that essentially justifies previously used "alternating maximization and minimization procedures" for the min-max problem. In the present treatment, no assumption of a saddle point solution is necessary, as long as the functions involved are once differentiable with respect to their arguments.

### 3.3 Numerical Examples

In this section, the first order algorithm is applied to solve four example problems from Section 2.3.

As a numerical example, for the weapons allocation problem given by Eqs. (2.16-19), the following input data from Medanic [8] are taken, with  $n=3$ :  $v_1 = 1, v_2 = 2, v_3 = 3, v_4 = 4$ ;  $\beta_1 = 1, \beta_2 = 2, \beta_3 = 2, \beta_4 = 1$ ;  $\gamma_1 = 4, \gamma_2 = 3, \gamma_3 = 2, \gamma_4 = 1$ .

Since  $f'$  in Eq. (2.16) is concave in  $\alpha$ , for any  $b$ , and the constrained set given by Eq. (2.18) is convex, only one value of  $\alpha$  is used as a starting estimate of the maximum point. The initial estimate is  $b^{(0)} = [0.1, 0.2, 0.3]^T$  and

$\alpha^{(0)} = [0.1, 0.2, 0.3]^T$ . Results of computation for each iteration are shown in Table 1. The final result is  $b = [0.0, 0.188, 0.273]^T$ ,  $\alpha = [0.0, 0.182, 0.456]^T$ , and  $J = -2.234$ , after 11 iterations. The solution from [8] is  $b = [0.0, 0.1891, 0.2942]^T$  and  $J = -2.235$ , after 22 iterations.

In Table 2, the results for case (i) of the 3 bar truss design problem are shown. At the optimum solution, the worst case was found to be the buckling constraint of member 2, at load angle  $\alpha = -45^\circ$ , and the  $z_1$ -displacement constraint, at load angle  $\alpha = 0.0^\circ$ . It is observed that the  $z_2$ -displacement constraint is nearly tight at load angle  $\alpha = 90^\circ$ . Table 3 shows results for case (ii). Two displacement constraints became tight at  $\alpha = 0^\circ$  and  $\alpha = 90^\circ$ , respectively. In Tables 2 and 3,  $\alpha$  is given in radians.

The results presented in Table 4, for case (i) of the vibration isolator design problem, show that the rattlespace constraint is strictly satisfied and that  $g_1$  is tight at two local maximum points. Thus, the solution of this problem is the same as Den Hartog's [30], with an infinite range of exciting frequency. The final result obtained is  $\xi = 0.1697$  and  $f = 0.9090$ , with  $x_1 = 4.59$ , after 20 iterations. The response of the main mass with this design is shown in Figure 6. The analytic solution from [30] is  $\xi = 0.1679$  and  $f = 0.9091$ , with  $x_1 = 4.583$ . The solution

in [31], using a grid mesh of 0.05, is  $\xi = 0.1503$  and  $f = 0.9096$ , with  $x_1 = 4.547$ . For the excitation frequency range of  $[0.9, 1.25]$  in case (ii), results are shown in Table 5. The final result is  $\xi = 0.1239$  and  $f = 0.9536$ , with  $x_1 = 4.06$ . Here,  $g_1$  becomes tight at two points, as shown in Figure 6, and the rattlespace constraint  $g_2$  is tight at  $\alpha = 0.94$ . The solution in [31], using a grid mesh of 0.05, is  $\xi = 0.1232$  and  $f = 0.9524$ , with  $x_1 = 4.029$ .

Table 6 shows results of calculation for the bridge design problem described in Section 2.3.5. The problem contains 9 design variables, 1 free parameter, 16 state variables, and 20 inner problems (14 subproblems; 8 stress constraints for the first eight truss members, 5 moment constraints at three nodal points and at two loads  $P_2$  and  $P_3$ , and 1 displacement constraint at load  $P_2$ ). The solution shows that the displacement constraint dominates the design. Convergence was rapid for the first few iterations, but the compensation of displacement violations was slow, resulting in a large number of iterations. Better convergence can be obtained by a different choice of the algorithm parameters, such as the stepsize. As in the other example problems, this problem shows the reliability of the first order method. The computer time per iteration on an IBM 360/65 was about 1.7 seconds in FORTRAN(H).

Table 1 Weapons Allocation Problem

(First Order Algorithm)

iter	$b_i$			obj.	$  \delta b^1  $	$a_i$			$  \delta a^1  $	tight $g_1, q_1$
						0.1	0.2	0.3		
0	0.1	0.2	0.3	-2.10	1.0	0.0	0.246	0.566	(4)*.007	$q_1$
1	0.15	0.154	0.254	-2.298	0.55	0.0	0.123	0.447	(4)0.008	$q_1$
2	-0.134	0.237	0.293	-2.378	0.17	0.0	0.219	0.455	(4)0.005	$g_2, q_1$
3	0.0	0.075	0.340	-2.280	0.65	0.0	0.010	0.478	(4)0.008	"
4	0.0	0.199	0.328	-2.308	0.43	0.0	0.197	0.556	(4)0.007	"
5	0.0	0.148	0.254	-2.274	0.20	0.0	0.147	0.506	(3)0.003	"
6	0.0	0.179	0.250	-2.240	0.16	0.0	0.160	0.445	(4)0.006	"
7	0.0	0.189	0.262	-2.232	0.09	0.0	0.166	0.466	(1)0.005	"
8	0.0	0.193	0.269	-2.234	0.04	0.0	0.178	0.452	(2)0.004	"
9	0.0	0.191	0.271	-2.234	0.04	0.0	0.180	0.453	(1)0.003	"
10	0.0	0.190	0.272	-2.234	0.04	0.0	0.181	0.454	(1)0.002	"
11	0.0	0.188	0.273	-2.234	0.04	0.0	0.182	0.456	(1)0.001	"

\*Number of maximizations.

Table 2 3-bar Truss Design, Case (1) (First Order Algorithm)

iter	$b_1$			obj.	$  \delta b^1  $	$  \delta \alpha^1  _{\max}$	tight constraint, $g_1(a)$
0	8.00	2.40	3.20	182.39	12.6	(6)*.300	$g_3, g_7, g_8$
1	8.03	2.42	4.61	203.01	11.6	(6)0.007	$g_3(-.61), g_7(-.18)$ $g_8(-.785), g_8(1.571)$
2	8.33	2.31	5.99	225.54	10.8	(6)0.007	$g_3(-.619), g_7(-.117)$ $g_8(1.571)$
3	8.84	2.09	7.29	249.00	3.75	(6)0.005	$g_7(-.072), g_8(1.571)$
4	9.28	1.55	8.49	266.54	2.07	(5)0.005	$g_7(-.045), g_8(1.571)$
5	9.67	1.18	9.26	279.52	0.95	(6)0.006	$g_7(-.013), g_8(1.571)$
6	10.07	0.894	9.77	289.54	0.60	(6)0.005	$g_7(-.0043), g_8(1.571)$
7	10.36	0.691	10.12	296.56	0.58	(6)0.003	$g_7(-.011), g_8(1.571)$
8	10.59	0.530	10.39	302.10	0.39	(5)0.005	$g_7(-.0018), g_8(1.571)$
9	10.83	0.369	10.66	307.58	0.41	(6)0.003	$g_7(-.007), g_8(1.571)$
10	11.02	0.237	10.88	312.04	0.25	(5)0.004	$g_7(.0002), g_8(1.571)$
11	11.21	0.105	11.09	316.50	0.33	(5)0.004	$g_7(-.008), g_8(1.571)$
12	11.35	0.00166	11.27	319.95	0.20	(5)0.003	$g_5(-.785), g_7(-.0016)$ $g_8(1.571)$
13	11.25	0.0445	11.38	320.42	0.30	(5)0.003	$g_5(-.785), g_7(.005)$
14	11.33	0.0448	11.30	320.43	0.08	(6)0.04	$g_5(-.785), g_7(-.0001)$
15	11.32	0.0448	11.31	320.45	0.14	(6)0.04	$g_5(-.785), g_7(-.005)$
16	11.306	0.04474	11.321	320.44	0.04	(6)0.03	$g_5(-.785), g_7(.0019)$
17	11.309	0.04475	11.319	320.45	0.1	(6)0.03	$g_5(-.785), g_7(-.005)$

Table 3 3-bar Truss Design, Case (11) (First Order Algorithm)

iter	$b_1$			obj.	$  \delta b^1  $	$  \delta a^1  _{\max}$	tight constraint, $g_1(a)$
0	8.00	2.40	3.20	182.39	97.8	(6)*.075	$g_3, g_7, g_8$
1	8.12	2.39	4.60	204.01	73.0	(6)0.0047	$g_3, g_7, g_8$
2	8.44	2.40	5.99	228.04	49.3	(6)0.0066	$g_7(0.0), g_8(1.571)$
3	8.91	2.38	7.32	253.35	29.9	(6)0.0046	$g_7(0.0), g_8(1.571)$
4	9.67	2.32	8.86	285.33	14.1	(4)0.0045	$g_7(0.0), g_8(1.571)$
5	10.85	2.15	10.71	326.42	2.23	(6)0.0060	$g_7(0.0), g_8(1.571)$
6	11.295	2.01	11.282	339.41	0.19	(5)0.0042	$g_7(0.0), g_8(1.571)$
7	11.313	2.00	11.314	340.00	0.015	(0)0.0042	$g_7(0.0), g_8(1.571)$
8	11.314	2.00	11.314	340.00	0.001	(0)0.0042	$g_7(0.0), g_8(1.571)$

\*Maximum number of maximizations



Table 4 Vibration Isolator Design, Case (i)

(First Order Algorithm)

iter	$b_1$		obj.	$  \delta b^1  $	$  \delta a^1  _{\max}$	tight constraint, $g(a)$
0	0.15	1.0	58.5	29.97	(6)*.0025	$g_1(.8675)$
1	0.15	0.9965	55.99	29.97	(5)0.0018	$g_1(.864)$
2	0.1502	0.9926	53.48	29.97	(6)0.0013	$g_1(.8665)$
3	0.1502	0.9889	50.98	29.97	(5)0.0018	$g_1(.863)$
4	0.1503	0.9844	48.49	29.96	(5)0.0013	$g_1(.8605)$
5	0.1504	0.9799	45.99	30.00	(5)0.0018	
6	0.1504	0.9799	40.00	29.95	(5)0.0025	$g_1(.859)$
7	0.1508	0.9636	37.52	29.92	(4)0.0025	$g_1(.854)$
8	0.1509	0.9564	35.03	30.00	(4)0.0025	
9	0.1509	0.9564	29.04	29.92	(5)0.0025	$g_1(.854)$
10	0.1512	0.9352	26.59	29.86	(5)0.0025	$g_1(.849)$
11	0.1513	0.9237	24.11	29.81	(4)0.0025	$g_1(.844)$
12	0.1514	0.9128	21.65	29.74	(5)0.0025	$g_1(.839)$
13	0.1515	0.9007	19.10	9.36	(4)0.0025	$g_1(.836), g_2(1.067)$
14	0.1739	0.9074	20.39	2.44	(5)0.0025	$g_1(.849), g_2(1.052)$
15	0.1658	0.9092	21.00	2.78	(4)0.0017	$g_1(.846), g_2(1.063)$
16	0.1720	0.9087	21.00	1.02	(4)0.0025	$g_1(.850), g_2(1.058)$
17	0.1706	0.9089	21.00	0.81	(4)0.0025	$g_1(.846), g_2(1.053)$
18	0.1701	0.9090	21.03	0.96	(4)0.0017	$g_1(.849), g_2(1.056)$
19	0.1699	0.9090	21.06	1.12	(3)0.0015	$g_1(.852), g_2(1.060)$
20	0.1697	0.9090	21.05	0.97	(4)0.0015	$g_1(.849), g_2(1.056)$



Table 5 Vibration Isolator Design, Case (ii)  
(First Order Algorithm)

iter	$b_1$		obj.	$  \delta b^1  $	$  \delta a^1  _{\max}$	tight constraint, $g_1(\alpha)$
0	0.15	1.0	37.82	28.24	(6)*.0025	$g_1(.9)$
1	0.1499	0.9955	35.55	28.24	(6)0.005	$g_1(.9)$
2	0.1498	0.9907	33.20	28.23	(6)0.0013	$g_1(.9)$
3	0.1497	0.9852	30.85	28.22	(4)0.5017	$g_1(.9)$
4	0.1496	0.9791	28.50	28.20	(4)0.5025	$g_1(.9)$
5	0.1494	0.9723	26.16	28.18	(4)0.0025	$g_1(.9)$
6	0.1491	0.9646	28.82	28.14	(4)0.0025	$g_1(.9)$
7	0.1488	0.9558	21.49	28.09	(5)0.0025	$g_1(.9)$
8	0.1484	0.9453	19.16	27.99	(4)0.0025	$g_1(.9)$
9	0.1478	0.9326	17.57	11.11	(4)0.0025	$g_1(.9), g_1(1.093)$
10	0.1297	0.9488	16.68	5.59	(5)0.0025	$g_1(.9), g_1(1.108)$
11	0.1168	0.9589	16.85	4.44	(5)0.0017	$g_1(.9), g_1(1.125)$
12	0.1262	0.9517	16.61	0.91	(4)0.0025	$g_1(.9), g_1(1.120)$
13	0.1311	0.9472	16.71	5.93	(4)0.0022	$g_1(.9), g_1(1.107)$
14	0.1150	0.9603	16.90	6.65	(3)0.0015	$g_1(.9), g_1(1.13), g_2(.96)$
15	0.1180	0.9587	16.67	4.47	(3)0.0015	$g_1(.9), g_1(1.13), g_2(.943)$
16	0.1192	0.9577	16.64	0.8(-4)	(4)0.0015	$g_1(.9), g_1(1.12), g_2(.94)$
17	0.1221	0.9552	16.62	0.93	(3)0.0013	$g_1(.9), g_1(1.12), g_2(.938)$
18	0.1227	0.9547	16.61	2.32	(4)0.0013	$g_1(.9), g_1(1.12), g_2(.94)$
19	0.1233	0.9541	16.61	0.88	(4)0.0013	$g_1(.9), g_1(1.12), g_2(.94)$
20	0.1239	0.9536	16.61	0.1(-4)	(4)0.0013	$g_1(.9), g_1(1.12), g_2(.94)$

Table 6 Bridge Design Problem  
(First Order Algorithm)

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Initial estimate;  $b^{(0)} = [3.8, 3.0, 3.0, 3.8, 0.8, 0.8, 3.0, 3.0, 1250.]^T$ ,

Weight = 12854 lbs,

Constraints violated; none.

Final design found:

$b^{(75)} = [3.228, 2.284, 1.107, 3.527, 1.847, 0.200, 2.340, 0.200, 1255.0]^T$ ,

Weight = 11825 lbs,

Displacement at load  $P_2 = 1.8029"$  at  $\alpha = 378.8"$  (0.16% violation),

Design variables  $b_6$  and  $b_8$  were at the lower bound of  $0.2 \text{ in}^2$ .

The following information regarding parametric constraints is obtained:

Maximum stresses in members 1 through 8 are;  $-18.9 \text{ ksi}$  at  $\alpha = 204"$ ,

$18.9 \text{ ksi}$  at  $204"$ ,  $34.6 \text{ ksi}$  at  $72.3"$ ,  $-16.4 \text{ ksi}$  at  $396"$ ,  $18.5 \text{ ksi}$  at  $570"$ ,

$18.5 \text{ ksi}$  at  $0.0"$ ,  $17.8 \text{ ksi}$  at  $234"$ , and  $7.22 \text{ ksi}$  at  $440"$ ;

Maximum moments were  $187.8 \text{ ft-kips}$  under load  $P_2$  at  $\alpha = 0"$  and  $187.7 \text{ ft-kips}$

under load  $P_3$  at  $785"$ .

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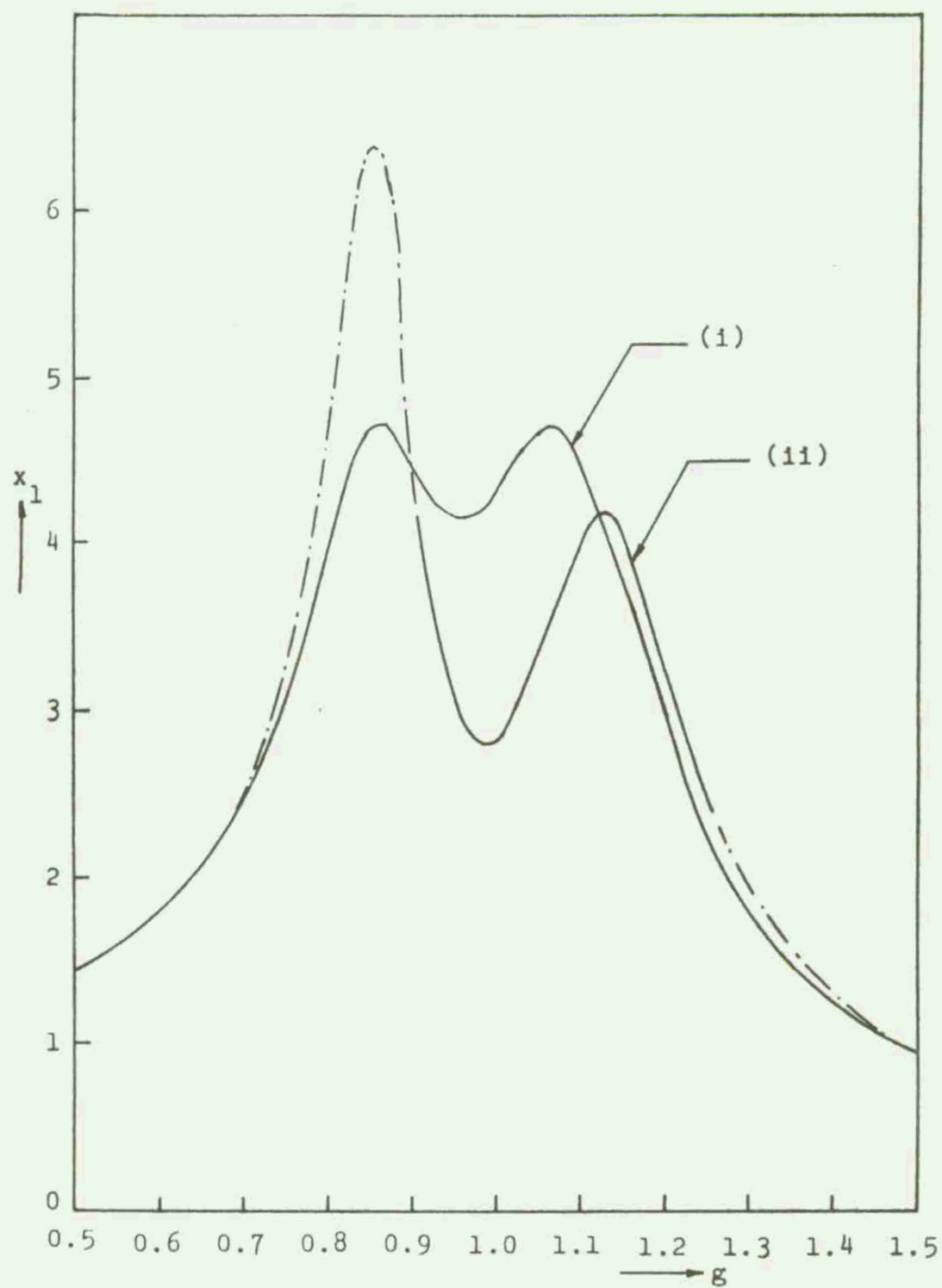


Figure 6 Main Mass Response

### 3.4 Conclusions

A first order algorithm has been developed, applying Kuhn-Tucker necessary conditions to the first order approximation of the POD problem. The analysis shows that this method is equivalent to alternating maximization and minimization procedures for min-max problems.

The algorithm implemented by the gradient projection method, with constraint error compensation, has been applied to four example problems with no real difficulty. It was experienced that, though the convergence is linear, the method has the general characteristics of the gradient projection method applied to nonlinear programming problems. Thus the method appears to be reliably convergent, generally within 20 iterations.

## APPENDICES

# APPENDIX A DYNAMIC EQUATIONS FOR SECTION 2.3.4

The dynamic equations have been derived in terms of nondimensionalized variables in [32]. With specific values for the variables introduced there (See Figure 3), the mass, stiffness and damping matrices are obtained as follows:

$$M = \begin{pmatrix} 0.25 & 0 & 0 & 0 & 0 \\ & 1.0 & 0 & 0 & 0 \\ & & 0.07463 & 0 & 0 \\ & & & 0.15 & 0 \\ & \text{(symmetric)} & & & 0.15 \end{pmatrix},$$

$$K = \begin{pmatrix} 0.410 & -0.410 & -0.03617 & 0 & 0 \\ & 2.590 & -0.3292 & -1.09 & -1.09 \\ & & 0.6084 & -0.3633 & 0.7267 \\ & & & 12.12 & 0 \\ & \text{(symmetric)} & & & 12.12 \end{pmatrix},$$

$$D = \begin{pmatrix} 0.6403b_1, & -0.6403b_1, & -0.05336b_1, & 0, & 0 \\ 0.6403b_1+2.697b_2+2.697b_3, & 0.05336b_1+.8989b_2-1.798b_3, & -2.697b_2, & -2.697b_3 \\ .004447b_1+.2996b_2+1.199b_3, & -.8989b_2, & 1.798b_3 \\ (symmetric) & & 2.697b_2, & 0 \\ & & & 2.697b_3 \end{pmatrix}$$



and the column force vector is given as

$$P = \{ 0, 0, 0, 11.03, 11.03 \},$$

where  $b_1$ ,  $b_2$ , and  $b_3$  are the dimensionless damping parameters, which are the design variables. Hence, the state equations can be written, from Eqs. (2.35) and (2.37), as follows:

$$\begin{pmatrix} -\alpha^2 M + K & \alpha D \\ \alpha D & \alpha^2 M - K \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} P \\ 0 \end{pmatrix}$$

where the dimensionless generalized coordinates

$z \equiv X/x_0 = u - iv$  are the state variables, and  $\alpha = \omega/\omega_{10}$  is the frequency parameter, where  $\omega_{10}$  is the first undamped natural frequency.

Therefore, the objective function introduced in the text as the square of the ratio of the amplitude of  $m_1$  and the base is given by

$$\phi(z) = u_1^2 + v_1^2.$$

## APPENDIX B

## ANALYSIS OF THE BRIDGE FOR SECTION 2.3.6

Numbering the elements and nodes of the truss structure, as shown in Figure 4a, extensions of the members, in terms of generalized displacements, are obtained: For convenience, let

$$A = \frac{a}{\sqrt{a^2 + h^2}}, \quad B = \frac{b}{\sqrt{b^2 + h^2}}$$

$$H_1 = \frac{h}{\sqrt{b^2 + h^2}}, \quad H_2 = \frac{h}{\sqrt{a^2 + h^2}}$$

Then,

$$\begin{aligned} e_1 &= Bu_2 - H_1 v_2, \\ e_2 &= u_3, \\ e_3 &= v_3 - v_2, \\ e_4 &= u_4 - u_2, \\ e_5 &= A(u_5 - u_2) + H_2(v_5 - v_2), \\ e_6 &= A(u_4 - u_3) + H_2(v_3 - v_4), \\ e_7 &= u_5 - u_3, \\ e_8 &= v_5 - v_4, \\ e_9 &= u_6 - u_4, \\ e_{10} &= A(u_7 - u_4) + H_2(v_7 - v_4), \\ e_{11} &= A(u_6 - u_5) + H_2(v_5 - v_6), \\ e_{12} &= u_7 - u_5, \end{aligned}$$

$$e_{13} = v_7 - v_6,$$

$$e_{14} = B(u_8 - u_6) - H_1 v_6,$$

$$e_{15} = u_8 - u_7.$$

The element stiffness matrices are, then, given by;

$$1. \quad \frac{A_1 E}{L_1} \begin{pmatrix} B^2 & -BH_1 \\ -BH_1 & H_1^2 \end{pmatrix}$$

$$2. \quad \frac{A_2 E}{L_2}$$

$$3. \quad \frac{A_3 E}{L_3} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$4. \quad \frac{A_4 E}{L_4} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$5. \quad \frac{A_5 E}{L_5} \begin{pmatrix} A^2 & AH_2 & -A^2 & -AH_2 \\ & H_2^2 & -AH_2 & -H_2^2 \\ & & A^2 & AH_2 \\ (sym) & & & H_2^2 \end{pmatrix}$$

$$6. \quad \frac{A_6 E}{L_6} \begin{pmatrix} A^2 & -AH_2 & -A^2 & AH_2 \\ & H_2^2 & AH_2 & -H_2^2 \\ & & A^2 & -AH_2 \\ (sym) & & & H_2^2 \end{pmatrix}$$

$$7. \quad \frac{A_7 E}{L_7} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \text{ etc.}$$

The structure stiffness matrix is obtained by combining the element stiffness matrices. This can be done easily by a simple computer program with the element stiffness matrices stored. The equation for the truss structure can be written as;

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \cdot \begin{bmatrix} q \\ q^b \end{bmatrix} = \begin{bmatrix} 0 \\ -F \end{bmatrix} \quad (\text{B.1})$$

where,

$$q \equiv \{u_2, v_2, u_3, u_4, v_4, u_5, u_6, v_6, u_7, u_8\}, \quad (\text{B.2})$$

$$q^b \equiv \{v_3, v_5, v_7\}, \quad (\text{B.3})$$

$$F \equiv \{F_3, F_5, F_7\}. \quad (\text{B.4})$$

For a more systematic approach of obtaining the stiffness matrix and the derivatives of the equations with respect to design variables, see [50].

Once the generalized displacements are obtained, axial stresses in the members are given by the formula

$$\sigma_1 = \frac{E}{L_1} e_1, \quad (\text{B.5})$$

where the subscript 1 refers the 1-th member.

The following analysis will determine the displacements at  $x = a_1, a_2$ , and  $a_3$ , in terms of the load  $P = \{P_1, P_2\}$  and the load  $F$  acting at the interacting boundary. Using the superposition principle, the displacements can be written as

$$EIq^b = BF + CP, \quad (\text{B.6})$$

where  $BF$  is the contribution from  $F$  and  $CP$  is that from

the load  $P$ .

Consider first the case of load  $P$ . Using the notation of singularity functions [49], the load on the beam is given by

$$q(x) = -R_A \langle x \rangle_{-1} + P_1 \langle x-a \rangle_{-1} + P_2 \langle x-a-l \rangle_{-1} - R_B \langle x-d \rangle_{-1}.$$

By integration, the shear force is,

$$-V(x) = -R_A \langle x \rangle^0 + P_1 \langle x-a \rangle^0 + P_2 \langle x-a-l \rangle^0 - R_B \langle x-d \rangle^0,$$

and integrating again, the moment is,

$$M(x) = -R_A \langle x \rangle^1 + P_1 \langle x-a \rangle^1 + P_2 \langle x-a-l \rangle^1. \quad (B.7)$$

From the load conditions at the end,

$$-V(x)|_{x=d+} = 0 = -R_A + P_1 + P_2 - R_B$$

$$M(x)|_{x=d+} = 0 = -R_A d + P_1(d-a) + P_2(d-a-l).$$

Hence solving for  $R_A$  and  $R_B$ ,

$$R_A = \{P_1(d-a) + P_2(d-a-l)\}/d, \quad (B.8)$$

$$R_B = \{P_1 a + P_2(a+l)\}/d. \quad (B.9)$$

Now, the beam equation is given by

$$EI \frac{d^2 v}{dx^2} = M(x) \quad (B.10)$$

where,  $v$  is the vertical displacement at  $x$ . After substituting  $M(x)$  from Eq. (B.7) and integrating,

$$EI v = \frac{1}{6} [-R_A x^3 + P_1 \langle x-a \rangle^3 + P_2 \langle x-a-l \rangle^3] + c_1 x + c_2. \quad (B.11)$$

From the boundary conditions, i.e.,

$$v|_{x=0} = 0, \quad \text{and} \quad v|_{x=d} = 0,$$

one has

$$\begin{aligned} c_1 &= -\frac{1}{6d}[-R_A d^3 + P_1(d-\alpha)^3 + P_2(d-\alpha-l)^3], \\ c_2 &= 0. \end{aligned} \quad (B.12)$$

Therefore, the vertical displacements  $v_3$ ,  $v_5$ , and  $v_7$ ; at  $x=a_1$ ,  $a_2$ , and  $a_3$ ; are obtained as

$$\begin{aligned} EIV_3 &= \frac{1}{6}[-R_A a_1^3 + P_1\langle a_1-\alpha \rangle^3 + P_2\langle a_1-\alpha-l \rangle^3] + c_1 a_1, \\ EIV_5 &= \frac{1}{6}[-R_A a_2^3 + P_1\langle a_2-\alpha \rangle^3 + P_2\langle a_2-\alpha-l \rangle^3] + c_1 a_2, \\ EIV_7 &= \frac{1}{6}[-R_A a_3^3 + P_1\langle a_3-\alpha \rangle^3 + P_2\langle a_3-\alpha-l \rangle^3] + c_1 a_3. \end{aligned}$$

The elements of  $C$  are obtained as:

$$\begin{aligned} C(i,j) &= \frac{1}{6d}[(d^2 a_i - a_i^3)(d-y_j) - a_i(d-y_j)^3 + d\langle a_i - y_j \rangle^3], \\ (i=1,2,3, \text{ and } j=1,2), \end{aligned} \quad (B.13)$$

where  $y_1=\alpha$ , and  $y_2=\alpha+l$ . In the case of  $k$  loads  $P_i$ ,  $i=1, \dots, k$ , it is only necessary to define  $y_j$  as positions of load applications.

For the load  $F_3$ ,  $F_5$ , and  $F_7$ , the matrix is simply obtained by putting  $\alpha=a_1$ ,  $a_2$ , and  $a_3$ , cyclically, into the first column of the above expression. It is noted that, in this case, Maxwell's law of reciprocity is applied and the matrix  $B$  is symmetric. The matrix  $B$  is obtained as:

$$\begin{aligned} B(i,j) &= \frac{1}{6d}[(d^2 a_i - a_i^3)(d-a_j) - a_i(d-a_j)^3], \\ (i,j=1,2,3; i \leq j). \end{aligned} \quad (B.14)$$

Since  $F$  is assumed as an unknown generalized force, one can write

$$\begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & I \\ 0 & I & -B/EI \end{bmatrix} \begin{bmatrix} q \\ q^b \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ CP/EI \end{bmatrix}, \quad (B.15)$$

which is the desired state equation, with the state variable vector

$$z = \{q, q^b, F\}. \quad (B.16)$$

Since B is a constant matrix, the following form is computationally more efficient;

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} + EIB^{-1} \end{bmatrix} \begin{bmatrix} q \\ q^b \end{bmatrix} = \begin{bmatrix} 0 \\ B^{-1}CP \end{bmatrix}, \quad (B.17)$$

$$F = EIB^{-1}q^b - B^{-1}CP. \quad (B.18)$$

In the text, bending moment and displacement of the beam are needed. For the beam under consideration, they are easily found as

$$M(x) = -R_A x + P_1 \langle x - \alpha \rangle^1 + P_2 \langle x - \alpha - l \rangle^1 + F_3 \langle x - a_1 \rangle^1 + F_5 \langle x - a_2 \rangle^1 + F_7 \langle x - a_3 \rangle^1, \quad (B.19)$$

and

$$EIv = \frac{1}{6}[-R_A x^3 + P_1 \langle x - \alpha \rangle^3 + P_2 \langle x - \alpha - l \rangle^3 + F_3 \langle x - a_1 \rangle^3 + F_5 \langle x - a_2 \rangle^3 + F_7 \langle x - a_3 \rangle^3] + c_1 x, \quad (B.20)$$

where

$$R_A = [P_1(d - \alpha) + P_2(d - \alpha - l) + F_3(d - a_1) + F_5(d - a_2) + F_7(d - a_3)]/d \quad (B.21)$$

$$c_1 = -[-R_A d^3 + P_1(d - \alpha)^3 + P_2(d - \alpha - l)^3 + F_3(d - a_1)^3 + F_5(d - a_2)^3 + F_7(d - a_3)^3]/6d. \quad (B.22)$$

It is noted that the maximum bending moment occurs at the point of load application. Hence the moment constraint is imposed at the discrete points of load application.

For example, at  $x = a_1$ , the constraint can be written



from Eq. (B.19)

$$|-R_A a_1 + P_1 \langle a_1 - a \rangle^1 + P_2 \langle a_1 - a - l \rangle^1| - 0.58 \sigma_Y I^{3/4} / SF \leq 0. \quad (B.23)$$

Displacement constraints will be imposed under a specified axle of the vehicle, say the front axle. Then, from Eq. (B.20),

$$|[-R_A a^3 + F_3 \langle a - a_1 \rangle^3 + F_5 \langle a - a_2 \rangle^3 + F_7 \langle a - a_3 \rangle^3] / 6 + c_1 a| - EI \delta \leq 0, \quad (B.24)$$

where  $\delta$  is a preassigned number. If the displacement constraint at the center is imposed, the expression is

$$|v_5| - \delta_c \leq 0, \quad (B.25)$$

where  $\delta_c$  is a constant.

For the numerical calculations given in the text, the following data are chosen:

$$E = 3 \cdot 10^7 \text{ lbs/in}^2$$

$$\rho = \rho_1 = 0.2836 \text{ lbs/in}^3$$

$$-\sigma_c = \sigma_T = \sigma_Y = 3.6 \cdot 10^4 \text{ lbs/in}^2$$

$$a = b = h = 30 \text{ ft}$$

$$l_1 = 14 \text{ ft}$$

$$l_2 = 42 \text{ ft}$$

$$P_1 = 8000 \text{ lbs}$$

$$P_2 = P_3 = 32000 \text{ lbs}$$

$$\delta = \delta_c = 1.8 \text{ in.}$$

APPENDIX C  
CONVEXITY AND SOME TERMINOLOGY FROM  
FUNCTIONAL ANALYSIS

A set  $K$  in a linear vector space is said to be convex, if, given  $x, y \in K$ , all points of the form  $\lambda x + (1-\lambda)y$ , with  $0 \leq \lambda \leq 1$ , are in  $K$ . The convex hull of a given subset  $S$  in a linear vector space is the smallest convex set containing  $S$ , and is denoted by  $\text{co } S$ . A set  $C$  in a linear vector space is said to be a cone with vertex at the origin if  $x \in C$  implies  $\lambda x \in C$  for all  $\lambda > 0$ .

For a full discussion of convexity, the reader is referred to [41].

Consider a normed linear space  $X$ . A linear functional  $f$  on  $X$  is bounded if there is a constant  $M$  such that  $|f(x)| \leq M||x||$ , for all  $x \in X$ . The normed dual of  $X$ , denoted by  $X^*$  is defined as the set of all bounded linear functionals on  $X$ . The elements of  $X^*$  are denoted by letters with an asterisk, e.g.,  $x^*, y^*$ , etc. The addition of functionals and multiplication by numbers are defined as follows:

$$(x^* + y^*)(x) = x^*(x) + y^*(x)$$

$$(\lambda x^*)(x) = \lambda x^*(x)$$

Then,  $X^*$  is a normed linear space (actually a Banach space)

with the norm

$$||x^*|| = \sup_{||x|| \leq 1} |x^*(x)|.$$

Hence the set

$$V(B, r, x_0^*) = \{x^* \in X^* \mid \sup_{x \in B} |x^*(x) - x_0^*(x)| < r\},$$

where  $B$  is any bounded subset of  $X$ , is an open set in the strong topology in  $X^*$ . The weak\* topology may be defined by the class of open sets;

$$S(x, r, x_0^*) = \{x^* \in X^* \mid |x^*(x) - x_0^*(x)| < r\},$$

for all  $x$ 's,  $x_0^*$ 's, and  $r$ 's. Therefore, on  $X^*$ , each term, such as closure, convergence, and compactness, has two meanings, depending on whether the strong or the weak\* topology is under consideration. It may be verified that the strong convergence of a sequence of functionals  $x_n^*$  to  $x_0^*$  implies weak\* convergence, and if a set is weak\* closed, then it is also strongly closed. For more details, refer to [6, 25, 51].

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